Supplementary material

The supplementary material contains:

- The asymptotic 'finite sample' approximations of the distribution of the test statistics.
- The auxiliary results which are used to define the tests in Sections 4 and 5.
- The simulations.
- The proof of the results.

6 Asymptotic 'finite sample' approximations of the distribution of the test statistics under the null

In this section we use the results in Section 4.4 to obtain finite sample asymptotic approximations to the distribution of the test statistics $\mathbf{T}_{1,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}')$, $\mathbf{T}_{2,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}')$ and $\mathbf{M}_{2,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}')$ under the null of stationarity.

Let $\{Z_{R,j}(\boldsymbol{r}_1,r_2), Z_{I,j}(\boldsymbol{r}_1,r_2); j=1,\ldots,T/2\}$ and $\{Z_{j,k}; j=1,\ldots,2|\mathcal{P}'|, k=1,\ldots,T/2\}$ denote iid standard Gaussian random variables (we use a double index because it simplifies some of the notations later on). We recall from the definition of $\widehat{V}_g(\omega_k;\mathcal{P}')$ in (29) that it is composed of $(2M+1)$-local average of $\Re\widehat{a}_g(\cdot)$ and $\Im\widehat{a}_g(\cdot)$, each term being asymptotically normal. Therefore we replace all the $\Re\widehat{a}_g(\cdot)$ and $\Im\widehat{a}_g(\cdot)$ in the definition of $\widehat{V}_g(\omega_k;\mathcal{P}')$ with standard normal distributions to give

$$\lambda^{d/2}\frac{\Re\widehat{a}_g(\omega_k;\boldsymbol{r}_1,r_2)}{\sqrt{\widehat{V}_g(\omega_k;\mathcal{P}')}}\sim t_{R,k}(\boldsymbol{r}_1,r_2) \quad \lambda^{d/2}\frac{\Im\widehat{a}_g(\omega_k;\boldsymbol{r}_1,r_2)}{\sqrt{\widehat{V}_g(\omega_k;\mathcal{P}')}}\sim t_{I,k}(\boldsymbol{r}_1,r_2),$$

where

$$\begin{aligned} t_{R,k}(\boldsymbol{r}_1,r_2) &= \lambda^{d/2}\frac{\Re\widehat{a}_g(\omega_k;\boldsymbol{r}_1,r_2)}{\sqrt{\widehat{V}_g(\omega_k;\mathcal{P}')}} \sim \frac{Z_{R,k}(\boldsymbol{r}_1,r_2)}{\sqrt{\frac{1}{2(2M+1)|\mathcal{P}'|-1}\sum_{i=-M}^{M}\sum_{j=1}^{2|\mathcal{P}'|}(Z_{j,k+i}-\bar{Z}_k)^2}} \\ t_{I,k}(\boldsymbol{r}_1,r_2) &= \lambda^{d/2}\frac{\Im\widehat{a}_g(\omega_k;\boldsymbol{r}_1,r_2)}{\sqrt{\widehat{V}_g(\omega_k;\mathcal{P}')}} \sim \frac{Z_{I,k}(\boldsymbol{r}_1,r_2)}{\sqrt{\frac{1}{2(2M+1)|\mathcal{P}'|-1}\sum_{i=-M}^{M}\sum_{j=1}^{2|\mathcal{P}'|}(Z_{j,k+i}-\bar{Z}_k)^2}} \end{aligned}$$

and $\bar{Z}_k=\frac{1}{2(2M+1)|\mathcal{P}'|}\sum_{i=-M}^{M}\sum_{j=1}^{2|\mathcal{P}'|}Z_{j,k+i}$ is the local average.

Noting that the test statistic is in terms of $\frac{\lambda^d T}{2}|\widehat{A}_{g,\widehat{V}^{-1/2}}(\boldsymbol{r}_1,r_2)|^2$, we replace the real and imaginary parts of $\widehat{a}_g(\omega_k;\boldsymbol{r}_1,r_2)/\sqrt{\widehat{V}_g(\omega_k;\mathcal{P}')}$ with the above to give

$$\frac{T\lambda^d}{2}\frac{|\widehat{A}_{g,\widehat{V}^{-1/2}}(\boldsymbol{r}_1,r_2)|^2}{V_{g,V^{-1/2}}} \sim X_{\boldsymbol{r}_1,r_2}$$

where

$$X_{\boldsymbol{r}_1,r_2} = \frac{T}{2}\left|\sum_{k=1}^{T/2} t_{R,k}(\boldsymbol{r}_1,r_2)\right|^2 + \frac{T}{2}\left|\sum_{k=1}^{T/2} t_{I,k}(\boldsymbol{r}_1,r_2)\right|^2,$$

for other $(\boldsymbol{r}_1,r_2)$, we use independent $\{Z_{k,R}(\boldsymbol{r}_1,r_2),Z_{k,I}(\boldsymbol{r}_1,r_2)\}$ but the same $\{Z_{j,k}\}$. We also recall that we estimate the variance $V_{g,V^{-1/2}}$, therefore we approximate its distribution with a weighted chi-squared with $(2|\mathcal{P}'|-1)$ degrees of freedom. Since the orthogonal sample which was used to estimate it contained $2|\mathcal{P}'|$ terms

$$\frac{\widehat{V}_{g,\widehat{V}^{-1/2}}}{V_{g,V^{-1/2}}} \sim \frac{1}{2|\mathcal{P}'|-1}\chi^2_{2|\mathcal{P}'|-1}.$$

Therefore, based on the above, the following distribution is used

$$\mathbf{T}_{1,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}') \sim \frac{1}{\frac{1}{2|\mathcal{P}'|-1}\chi^2_{2|\mathcal{P}'|-1}}\sum_{(\boldsymbol{r}_1,r_2)\in\mathcal{S}_1} X_{\boldsymbol{r}_1,r_2}$$

to approximate the 'asymptotic finite sample properties' of $\mathbf{T}_{1,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}')$, under the null. Using the same method we can obtain the 'asymptotic finite sample properties' for $\mathbf{M}_{1,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}')$.

Next we consider the local average DFTs, $\widehat{B}_{g,\widehat{V}^{-1/2}}(\omega_{jH};\boldsymbol{r}_1,r_2)$, $\widehat{D}_{g,\widehat{V}^{-1/2},\widehat{W};H}(\boldsymbol{r}_1,r_2)$ and $\widehat{D}_{g,\widehat{V}^{-1/2},1;H}(\boldsymbol{r}_1,r_2)$, which lead to the test statistics $\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$ and $\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$. Using the arguments given above we have,

$$H\lambda^d\left|\widehat{B}_{g,\widehat{V}^{-1/2};H}(\omega_{jH};\boldsymbol{r}_1,r_2)\right|^2 \sim Y_{jH}(\boldsymbol{r}_1,r_2),$$

where

$$Y_{jH}(\boldsymbol{r}_1,r_2) = \left|\frac{1}{\sqrt{H}}\sum_{k=1}^{H} t_{R,jH+k}(\boldsymbol{r}_1,r_2)\right|^2 + \left|\frac{1}{\sqrt{H}}\sum_{k=1}^{H} t_{I,jH+k}(\boldsymbol{r}_1,r_2)\right|^2.$$

Using the above, we estimate the distribution of $\widehat{D}_{g,\widehat{V}^{-1/2},1;H}(\boldsymbol{r}_1,r_2)$ with

$$H\lambda^d\widehat{D}_{g,\widehat{V}^{-1/2},1;H}(\boldsymbol{r}_1,r_2) \sim \frac{2H}{T}\sum_{j=1}^{T/2H} Y_{jH}(\boldsymbol{r}_1,r_2).$$

This gives us the 'asymptotic finite sample' distributions of $\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}(\mathcal{P},\mathcal{P}')$ and $\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}(\mathcal{P},\mathcal{P}')$.

In order to derive the sampling properties of $\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}(\mathcal{P},\mathcal{P}')$ and $\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}(\mathcal{P},\mathcal{P}')$, we recall that $\widehat{D}_{g,\widehat{V}^{-1/2},\widehat{W};H}(\boldsymbol{r}_1,r_2)$ involves $\widehat{W}_{g,\widehat{V}^{-1/2}}(\omega_{jk};\mathcal{P}')$ and we approximate this distribution by independent chi-squares $\{\frac{1}{2|\mathcal{P}'|-1}\chi^2_{2|\mathcal{P}'|-1,k}\}_{k=1}^{T/2H}$. This gives

$$\lambda^d H\widehat{D}_{g,\widehat{V}^{-1/2},\widehat{W};H}(\boldsymbol{r}_1,r_2)\sim\frac{2H}{T}\sum_{j=1}^{T/2H}\frac{Y_{jH}(\boldsymbol{r}_1,r_2)}{\frac{1}{2|\mathcal{P}'|-1}\chi^2_{2|\mathcal{P}'|-1,k}}.$$

Using this we can obtain the distributions of $\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}(\mathcal{P},\mathcal{P}')$ and $\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}(\mathcal{P},\mathcal{P}')$. Note that the same $\{\frac{1}{2|\mathcal{P}'|-1}\chi^2_{2|\mathcal{P}'|-1,k}\}_{k=1}^{T/2H}$ is used for all $\{\widehat{D}_{g,\widehat{V}^{-1/2},\widehat{W};H}(\boldsymbol{r}_1,r_2);(\boldsymbol{r}_1,r_2)\in\mathcal{P}\}$.

The 'asymptotic finite sample' distribution derived above are used in all the simulations below.

7 Auxiliary Results

7.1 Results in the case of stationarity

We first consider the sampling properties of $\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2)$, which is used to define the test statistics $\mathbf{T}_{1,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}')$ and $\mathbf{M}_{1,g,\widehat{V}^{-1/2}}(\mathcal{P},\mathcal{P}')$.

Lemma 7.1 *Suppose Assumptions 3.1 and 4.1 hold,* $0\le r_2,r_4\le T/2-1$, $\boldsymbol{r}_1\neq-\boldsymbol{r}_3$ *and* $h:[0,\pi]\to\mathbb{R}$ *is a Lipschitz continuous function. Then*

$$\begin{aligned}\frac{\lambda^dT}{2}\mathrm{cov}\left[\Re\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2),\Re\widehat{A}_{g,h}(\boldsymbol{r}_3,r_4)\right] &= I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2})+O\Big(\ell_{\lambda,a,n}+T^{-1}\Big) \quad (42)\\ \frac{\lambda^dT}{2}\mathrm{cov}\left[\Im\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2),\Im\widehat{A}_{g,h}(\boldsymbol{r}_3,r_4)\right] &= I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2})+O\Big(\ell_{\lambda,a,n}+T^{-1}\Big)\end{aligned}$$

and $\lambda^dT\mathrm{cov}[\Re\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2),\Im\widehat{A}_{g,h}(\boldsymbol{r}_3,r_4)]=O(\ell_{\lambda,a,n}+T^{-1})$ *where,*

$$\begin{aligned}V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2}) &= \frac{1}{\pi}\int_0^\pi|h(\omega)|^2V_g(\omega;\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2})d\omega+\frac{2}{(2\pi)^d\pi^2}\int_0^\pi\int_0^\pi\int_{\mathcal{D}^2}g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)}h(\omega_1)\overline{h(\omega_2)}\\ &\quad\times f_4(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_1+\omega_{r_2},\boldsymbol{\Omega}_2,\omega_2,-\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1},-\omega_2-\omega_{r_2})d\boldsymbol{\Omega}_1d\boldsymbol{\Omega}_2d\omega_1d\omega_2.\end{aligned}$$

Let $\{(\boldsymbol{r}_j,r_j);1\le j\le m\}$ *be a collection of integer vectors constrained such that* $0\le r_j\le T/2-1$ *and* $\boldsymbol{r}_{j_1}\neq-\boldsymbol{r}_{j_2}$. *Then under stationarity of* $\{Z_t(\boldsymbol{s})\}$ *and sufficient mixing conditions we have,*

$$\sqrt{\frac{\lambda^dT}{2}}\left[\frac{\Re\widehat{A}_{g,h}(\boldsymbol{r}_{j_1},r_{j_1})}{V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_{j_1}},\omega_{r_{j_1}})^{1/2}},\frac{\Im\widehat{A}_{g,h}(\boldsymbol{r}_{j_1},r_{j_1})}{V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_{j_1}},\omega_{r_{j_1}})^{1/2}},\ldots,\frac{\Re\widehat{A}_{g,h}(\boldsymbol{r}_{j_m},r_{j_m})}{V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_{j_m}},\omega_{r_{j_m}})^{1/2}},\frac{\Im\widehat{A}_{g,h}(\boldsymbol{r}_{j_m},r_{j_m})}{V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_{j_m}},\omega_{r_{j_m}})^{1/2}}\right]\\ \stackrel{\mathcal{D}}{\to}\mathcal{N}\left(0,I_{2m}\right).$$

We note that when $\|\boldsymbol{r}_1\|_1 << \lambda$ and $|r_2| << T$ that the variances above approximate to

$$V_{g,h}(\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2}) = V_{g,h}(0,0) + O\left(\frac{\|\boldsymbol{r}_1\|_1}{\lambda} + \frac{|r_2|}{T}\right).$$

We now consider the sampling properties of $\widehat{B}_{g,h;H}(\boldsymbol{r}_1, r_2)$ and $\widehat{D}_{g,h,v;H}(\boldsymbol{r}_1, r_2)$, which are used to define the test statistics $\mathbf{T}_{2,g,\widehat{V}^{-1/2},W}(\mathcal{P},\mathcal{P}')$ and $\mathbf{M}_{2,g,\widehat{V}^{-1/2},W}(\mathcal{P},\mathcal{P}')$. We start by studying $\widehat{B}_{g,h;H}(\boldsymbol{r}_1, r_2)$.

Lemma 7.2 *Suppose Assumptions 3.1 and 4.1 hold, $0 \leq r_2, r_4 \leq T/2 - 1$, $\boldsymbol{r}_1 \neq -\boldsymbol{r}_3$ and $h : [0,\pi] \to \mathbb{R}$ is a Lipschitz continuous function. Then*

$$\lambda^d H \mathrm{cov}\left[\Re\widehat{B}_{g,h;H}(\omega_{jH};\boldsymbol{r}_1, r_2), \Re\widehat{B}_{g,h;H}(\omega_{jH};\boldsymbol{r}_3, r_4)\right]$$
$$= I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4} W_{g,h}(\omega_{jH};\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2}) + O\Big(\ell_{\lambda,a,n} + H^{-1}\Big). \quad (43)$$

Exactly the same result holds for $\lambda^d H \mathrm{cov}\left[\Im\widehat{B}_{g,h;H}(\omega_{jH};\boldsymbol{r}_1, r_2), \Im\widehat{B}_{g,h;H}(\omega_{jH};\boldsymbol{r}_3, r_4)\right]$, where

$$2W_{g,h}(\omega_{jH};\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2})$$
$$= \frac{T}{H\pi}\int_{\omega_{jH}}^{\omega_{(j+1)H}} |h(\omega)|^2 V_g(\omega;\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2})d\omega + \frac{T}{H(2\pi)^{2d+2}}\int_{\omega_{jH}}^{\omega_{(j+1)H}}\int_{\omega_{jH}}^{\omega_{(j+1)H}}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)}$$
$$\times h(\omega_1)\overline{h(\omega_2)} f_4(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_{r_1}, \omega_1 + \omega_{r_2}, \boldsymbol{\Omega}_2, \omega_2, -\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_{\boldsymbol{r}_1}, -\omega_2 - \omega_{r_2}) d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 d\omega_1 d\omega_2,$$

noting that the first (covariance) term in $W_{g,h} = O(1)$, whereas the second term of $W_{g,h}$ which is the fourth order cumulant term is of order $O(H/T)$ since the cumulant term involves a double integral which is of order $O((H/T)^2)$. On the other hand, if $(\boldsymbol{r}_1, r_2), (\boldsymbol{r}_3, r_4) \neq 0$, then (with $0 \leq r_2, r_4 < T/2$ and $(\boldsymbol{r}_1, r_2) \neq -(\boldsymbol{r}_3, r_4)$) we have $\lambda^d H \mathrm{cov}\left[\Re\widehat{B}_{g,h;H}(\omega_{jH};\boldsymbol{r}_1, r_2), \Im\widehat{B}_{g,h;H}(\omega_{jH};\boldsymbol{r}_3, r_4)\right] = O(\ell_{\lambda,a,n})$ and for $j_1 \neq j_2$,

$$\lambda^d H \mathrm{cov}\left[\Re\widehat{B}_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1, r_2), \Re\widehat{B}_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_3, r_4)\right] = O\Big(\ell_{\lambda,a,n} + H^{-1} + \frac{H}{T}\Big), \quad (44)$$

where the same holds for $\lambda^d H \mathrm{cov}\left[\Im\widehat{B}_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1, r_2), \Im\widehat{B}_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_3, r_4)\right]$ and also for $\lambda^d H \mathrm{cov}\left[\Re\widehat{B}_g\right.$

Let $\{(k_j, \boldsymbol{r}_{1,i}, r_{2,i}); 1 \leq j \leq m, (\boldsymbol{r}_1, r_2) \in \mathcal{P}\}$ be a collection of integer vectors constrained such that, $1 \leq k_j \leq T/2$, $1 \leq r_j \leq T/2$ and $\boldsymbol{r}_{j_1} \neq -\boldsymbol{r}_{j_2}$. Then under stationarity of $\{Z_t(\boldsymbol{s})\}$ and sufficient mixing conditions we have,

$$\sqrt{\lambda^d H}\left[\frac{\Re\widehat{B}_{g,h}(\omega_{k_jH};\boldsymbol{r}_{1,i}, r_{2,i})}{W_{g,h}(\omega_{k_jH})^{1/2}}, \frac{\Im\widehat{B}_{g,h}(\omega_{k_jH};\boldsymbol{r}_{1,i}, r_{2,i})}{W_{g,h}(\omega_{k_jH})^{1/2}}, 1 \leq j \leq m, (\boldsymbol{r}_{1,i}, r_{2,i}) \in \mathcal{P}\right]$$
$$\overset{\mathcal{D}}{\to} \mathcal{N}\left(0, I_{2m|\mathcal{P}|}\right),$$

where $W_{g,h}(\omega) = W_{g,h}(\omega; 0, 0)$.

In the following lemma we consider the sampling properties of $\widehat{D}_{g,h,v;H}(\boldsymbol{r}_1, r_2)$. Note that we consider general functions v, whereas in Section 4.3 we set v to be the variance of $W_{g,h}(\omega)$, which means the mean of $\widehat{D}$ is asymptotically pivotal.

Lemma 7.3 *Suppose the assumptions in Lemma 7.2 hold and* $h : [0, \pi] \to \mathbb{R}$ *is a Lipschitz continuous function. Then we have*

$$\begin{aligned} & \mathrm{E}[H\lambda^d \widehat{D}_{g,h,v;H}(\boldsymbol{r}_1, r_2)] \\ = \; & E_{g,h,v}(\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2}) + O\left(\ell_{\lambda,a,n} + \frac{1}{H} + \frac{H}{T} + \frac{\lambda^d H[\prod_{j=1}^{d-b}(\log\lambda + \log|m_j|)]^2}{(T^{I_{r_2-r_4\neq 0}}\lambda^{d-b})^2}\right) \end{aligned} \tag{45}$$

and

$$\begin{aligned} & \frac{T}{2H}\mathrm{cov}\left[\lambda^d H\widehat{D}_{g,h,v;H}(\boldsymbol{r}_1, r_2), \lambda^d H\widehat{D}_{g,h,v;H}(\boldsymbol{r}_3, r_4)\right] \\ = \; & \begin{cases} U_{g,h,v}(\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2}) + O\left(\frac{H}{T} + \ell_{\lambda,a,n}\right) & \boldsymbol{r}_1 = \boldsymbol{r}_3 \text{ and } r_2 = r_4 \\ O\left(\frac{H}{T} + \ell_{\lambda,a,n}\right) & \text{otherwise} \end{cases} \end{aligned} \tag{46}$$

where,

$$\begin{aligned} E_{g,h,v}(\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2}) &= \frac{1}{\pi}\int_0^{\pi} \frac{W_{g,h}(\omega; \boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2})}{v(\omega)} d\omega \\ \text{and } U_{g,h,v}(\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2}) &= \frac{1}{\pi}\int_0^{\pi} \frac{|W_{g,h}(\omega; \boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2})|^2}{|v(\omega)|^2} d\omega. \end{aligned}$$

Let $\{(\boldsymbol{r}_j, r_j); 1 \leq j \leq m\}$ *be a collection of integer vectors constrained such that* $1 \leq k_j \leq T/2$, $1 \leq r_j \leq T/2$ *and* $\boldsymbol{r}_{j_1} \neq -\boldsymbol{r}_{j_2}$, *then we have*

$$\sqrt{\frac{T}{2HU_{g,h,v}}}\begin{pmatrix} \lambda^d H\widehat{D}_{g,h,v;H}(\boldsymbol{r}_{j_1}, r_{j_1}) - E_{g,h,v} \\ \vdots \\ \lambda^d H\widehat{D}_{g,h,v;H}(\boldsymbol{r}_{j_m}, r_{j_m}) - E_{g,h,v} \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, I_{2m})$$

where $U_{g,h,v} = U_{g,h,v}(0,0)$ *and* $E_{g,h,v} = E_{g,h,v}(0,0)$.

7.2 Results in the case of nonstationarity

We first generalize (14) from covariances to fourth order cumulants. We assume there exists a function κ such that

$$\begin{aligned}&\mathrm{cov}[Z_{t,\lambda,T}(\boldsymbol{s}), Z_{t+h_1,\lambda,T}(\boldsymbol{s}+\boldsymbol{v}_1), Z_{t+h_2,\lambda,T}(\boldsymbol{s}+\boldsymbol{v}_2), Z_{t+h_3,\lambda,T}(\boldsymbol{s}+\boldsymbol{v}_3)]\\ &\quad= \kappa_{h_1,h_2,h_3;\frac{t}{T}}\left(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3;\frac{\boldsymbol{s}}{\lambda}\right) + O\left(\frac{\prod_{i=1}^3 \beta_{2+\delta}(\boldsymbol{v}_i)\rho_{h_i}}{T}\right),\end{aligned} \tag{47}$$

where, $\sup_{u,\boldsymbol{s}} |\kappa_{h_1,h_2,h_3;u}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3;\boldsymbol{s})| \leq \prod_{i=1}^3 \rho_{h_i}\beta_{2+\delta}(\boldsymbol{v}_i)$,

$$\begin{aligned}|\kappa_{h_1,h_2,h_3;u_1}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3;\boldsymbol{s}) - \kappa_{h_1,h_2,h_3;u_2}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3;\boldsymbol{s})| &\leq |u_1-u_2|\prod_{i=1}^3 \beta_{2+\delta}(\boldsymbol{v}_i)\rho_{h_i},\\ |\kappa_{h_1,h_2,h_3;u}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3;\boldsymbol{s}_1) - \kappa_{h_1,h_2,h_3;u}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3;\boldsymbol{s}_2)| &\leq |\boldsymbol{s}_1-\boldsymbol{s}_2|_1\prod_{i=1}^3 \beta_{2+\delta}(\boldsymbol{v}_i)\rho_{h_i}.\end{aligned}$$

Using the above, we define the location and time dependent fourth order spectral density as

$$\begin{aligned}&F_{u,4}(\boldsymbol{\Omega}_1,\omega_1,\boldsymbol{\Omega}_2,\omega_2,\boldsymbol{\Omega}_3,\omega_3;\boldsymbol{s})\\ &\quad= \frac{1}{(2\pi)^3}\sum_{h_1,h_2,h_3\in\mathbb{Z}} e^{-i(h_1\omega_1+h_2\omega_2+h_3\omega_3)}\int_{\mathbb{R}^{3d}} \kappa_{h_1,h_2,h_3;u}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3;\boldsymbol{s})\\ &\qquad\qquad\times e^{-i(\boldsymbol{v}_1'\boldsymbol{\Omega}_1+\boldsymbol{v}_2'\boldsymbol{\Omega}_2+\boldsymbol{v}_3'\boldsymbol{\Omega}_3)}d\boldsymbol{v}_1 d\boldsymbol{v}_2 d\boldsymbol{v}_3.\end{aligned} \tag{48}$$

In order to obtain the expressions below we start by generalizing the covariance result in (16) to fourth order cumulants. By using Lee and Subba Rao [2015] and similar methods to those used in Bandyopadhyay and Subba Rao [2016] it can be shown that

$$\begin{aligned}&\mathrm{cum}[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1},\omega_{k_2}), \overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2})}, \overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_3},\omega_{k_4})}, J(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3},\omega_{k_4+r_4})]\\ &= \frac{1}{T\lambda^d} b_{r_2-r_4,4}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2},\boldsymbol{\Omega}_{\boldsymbol{k}_3},\omega_{k_4},\boldsymbol{\Omega}_{-\boldsymbol{k}_3-\boldsymbol{r}_3},\omega_{-k_4-r_4};\boldsymbol{r}_1-\boldsymbol{r}_3) + O\left(\frac{1}{T^2\lambda^d}+\frac{1}{T\lambda^{d+1}}\right)\end{aligned}$$

where,

$$\begin{aligned}&b_{r_2,4}(\boldsymbol{\Omega}_1,\omega_1,\boldsymbol{\Omega}_2,\omega_2,\boldsymbol{\Omega}_3,\omega_3;\boldsymbol{r}_1)\\ &\quad= \int_0^1\int_{[-1/2,1/2]^d} F_{u,4}(\boldsymbol{\Omega}_1,\omega_1,\boldsymbol{\Omega}_2,\omega_2,\boldsymbol{\Omega}_3,\omega_3;\boldsymbol{s})\, e^{-2\pi i\boldsymbol{r}_1'\boldsymbol{s}} e^{-2\pi i r_2 u} d\boldsymbol{s}\, du,\end{aligned}$$

and $F_{u,4}$ is defined in (48).

Lemma 7.4 *Suppose the assumptions in Assumptions 3.1 and 4.1 (generalized to the nonsta-*

tionary set-up) and (47) are satisfied. Then we have,

$$\lambda^d \text{cov}\left[\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2),\widehat{a}_g(\omega_{k_4};\boldsymbol{r}_3,r_4)\right] = b^{(1)}_{r_2,r_4,k_2,k_4}(\omega_{k_2},\omega_{k_4};\boldsymbol{r}_1,\boldsymbol{r}_3) + O\left(\frac{1}{T}+\frac{1}{\lambda}\right),$$

and

$$\lambda^d \text{cov}\left[\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2),\overline{\widehat{a}_g(\omega_{k_4};\boldsymbol{r}_3,r_4)}\right] = b^{(2)}_{r_2,r_4,k_2,k_4}(\omega_{k_2},\omega_{k_4};\boldsymbol{r}_1,\boldsymbol{r}_3) + O\left(\frac{1}{T}+\frac{1}{\lambda}+\ell_{\lambda,a,n}\right).$$

We also have,

$$\begin{aligned}
&\frac{T\lambda^d}{2}\text{cov}\left[\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2),\widehat{A}_{g,h}(\boldsymbol{r}_3,r_4)\right] \\
&= \frac{2}{T\lambda^d}\sum_{k_2,k_4=1}^{T/2} h(\omega_{k_2})\overline{h(\omega_{k_4})}b^{(1)}_{r_2,r_4,k_2,k_4}(\omega_{k_2},\omega_{k_4};\boldsymbol{r}_1,\boldsymbol{r}_3) \\
&\quad + \frac{2}{T^2\lambda^{2d}}\sum_{k_2,k_4=1}^{T/2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} h(\omega_{k_2})\overline{h(\omega_{k_4})}g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})} \\
&\quad \times b_{r_2-r_4,4}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2},\boldsymbol{\Omega}_{\boldsymbol{k}_3},\omega_{k_4},\boldsymbol{\Omega}_{-\boldsymbol{k}_3-\boldsymbol{r}_2},\omega_{-k_4-r_4};\boldsymbol{r}_1-\boldsymbol{r}_3) + O\left(\frac{1}{T}+\frac{1}{\lambda}+\ell_{\lambda,a,n}\right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{T\lambda^d}{2}\text{cov}\left[\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2),\overline{\widehat{A}_{g,h}(\boldsymbol{r}_3,r_4)}\right] \\
&= \frac{2}{T\lambda^d}\sum_{k_2,k_4=1}^{T/2} h(\omega_{k_2})h(\omega_{k_4})b^{(2)}_{r_2,r_4,k_2,k_4}(\omega_{k_2},\omega_{k_4};\boldsymbol{r}_1,\boldsymbol{r}_3) \\
&\quad + \frac{2}{T^2\lambda^{2d}}\sum_{k_2,k_4=1}^{T/2}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} h(\omega_{k_2})h(\omega_{k_4})g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})g(\boldsymbol{\Omega}_{\boldsymbol{k}_3}) \\
&\quad \times b_{r_2+r_4,4}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2},\boldsymbol{\Omega}_{-\boldsymbol{k}_3},\omega_{-k_3},\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3},\omega_{k_4+r_4};\boldsymbol{r}_1+\boldsymbol{r}_3) + O\left(\frac{1}{T}+\frac{1}{\lambda}+\ell_{\lambda,a,n}\right).
\end{aligned}$$

8 Simulations

8.1 Set-up

We now assess the finite sample performances of the test statistics described above through simulations. In all cases we consider mean zero spatio-temporal processes, where $T = 200$ and at each time point we observe $n = 100$ or 500 locations (the locations are drawn from a uniform distribution defined on $[-\lambda/2,\lambda/2]^2$ and we use the same set of locations at each time point). All tests are done at the 5% level and all results are based on 300 replications. Further, we investigate the performances of the tests when the coefficients $\widehat{a}_g(\omega_k; \boldsymbol{r}_1, r_2)$, as defined in Section 4.1, are being calculated while both removing and keeping the 'nugget effect' N_T (in Tables 1-6 the rejection rates for the test statistics without removing N_T are reported in the parentheses). All simulations are done for spatial dimension $d = 2$.

Next, we briefly discuss the implementation issues.

1. *Choice of set $\mathcal{P}$ and $\mathcal{P}'$*: All test statistics depend on the choice of $\mathcal{P}$ and $\mathcal{P}'$. In all simulations described in this section we use $\mathcal{P} = \{(1,0),(1,1),(0,1),(-1,1)\} \times \{1,2\}$ and $\mathcal{P}' = \{(1,0),(1,1),(0,1),(-1,1)\} \times \{4,5\}$ to calculate empirical type I errors and overall powers. Further, to test for stationarity over space, we take $\mathcal{P} = \{(1,0),(1,1),(0,1),(-1,1)\} \times \{0\}$ and $\mathcal{P}' = \{(2,0),(2,1),(2,2),(1,2),(0,2),(-1,2),$
$(-2,2),(-2,1)\} \times \{0\}$ and to test for stationarity over time, we take $\mathcal{P} = \{(0,0)\} \times \{1,2\}$ and $\mathcal{P}' = \{(0,0)\} \times \{4,5\}$.

2. *Choice of $g(\cdot)$* : Based on the discussion in Remark 2.1 we use the weight function $g(\boldsymbol{\Omega}) = \sum_{j=1}^{L} e^{i\boldsymbol{v}_j'\boldsymbol{\Omega}}$. The choice of $\boldsymbol{v}_j$'s should depend on the density of the sampling region. Following the same rationale as described in Bandyopadhyay and Subba Rao [2016], in all simulations we define the v grid as $\mathbb{V} = \{\boldsymbol{v}_j = (v_{j1}, v_{j2})' \in \mathbb{R}^d : v_{jk} = -s, -s/2, 0, s/2, s, \text{ for } k = 1,2\}$ such that $\boldsymbol{v}_j + \boldsymbol{v}_{j'} \neq \boldsymbol{0}$ for $\boldsymbol{v}_j, \boldsymbol{v}_{j'} \in \mathbb{V}$, where $s = \lambda/n^{1/d}$ is the 'average spacing' between the observations on each axis. We should mention that if the support of the empirical covariance of the data appears far greater than $s = \lambda/n^{1/d}$, then using a wider v grid is appropriate. If changes in the spatial covariance function happen mainly at lags much smaller than $s = \lambda/n^{1/d}$, then data is not available to detect changes in the spatial covariance structure.

3. *Choice of frequency grid*: In all simulations we use $a = \sqrt{n}$ in the definition of $\widehat{a}_g(\omega_k; \boldsymbol{r}_1, r_2)$.

4. *Choice of H in the definition of $\mathbf{T}_2$ and $\mathbf{M}_2$*: For all simulations we use $H = 10$ and $H = 20$.

5. *Choice of M to calculate the local averages*: In order to estimate $V_g(\omega_k)$ we use the estimator $\widehat{V}_g(\omega_k; \mathcal{P}')$ (defined in (29)) with $M = 2$ (thus taking a local average of 5).

To obtain the critical values of the tests we use the asymptotic 'finite sample' approximations of the distributions of the test statistics as described in Section 4.4. For ease of discussion below we refer to (i) $\mathbf{T}_{1,g,\widehat{V}^{-1/2}}$ and $\mathbf{M}_{1,g,\widehat{V}^{-1/2}}$ as the average covariance test statistics (ii) $\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$ and $\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$ as the average squared covariance test statistics and (iii) $\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$ and $\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$ as the variance adjusted average squared covariance test statistics.

8.2 Simulations under the null

8.2.1 Models

In order to define the spatio-temporal models, we start by defining the 'innovations' process. Let $\{\varepsilon_t(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^2, t \in \mathbb{Z}\}$ denote a spatio-temporal stationary Gaussian random field which is independent over time with spatial exponential covariance $\text{cov}[\varepsilon_t(\boldsymbol{s}_1), \varepsilon_t(\boldsymbol{s}_2)] = \exp(-\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|_2/\rho)$, where ρ is the 'range parameter'. We do all the simulations under the null with $\rho = 0.5$, $\rho = 1$ and $\lambda = 5$ (in the case that $\rho = 1$ the range of dependence for the innovations is 20%, whereas for $\rho = 0.5$ the range reduces to 10%; see Figure 1). We mention that the decorrelation property of Fourier Transforms, given in Lemma 3.1 implicitly depends on the range of dependence with respect to λ. If the range of dependence is too large with respect to the observed random field then the degree of correlation in the Fourier transforms will be non-negligible (leading to false rejection of the null).

(S1) *Spatially and temporally stationary Gaussian random field*: We define a spatio-temporal model with the temporal AR(1) structure $Z_t(\boldsymbol{s}) = 0.5Z_{t-1}(\boldsymbol{s}) + \varepsilon_t(\boldsymbol{s})$.

(S2) *Spatially and temporally stationary non-Gaussian random field:* To induce non-linearity and non-Gaussianity in the random field we use a Bilinear model of the form

$$Z_t(\boldsymbol{s}) = 0.5Z_{t-1}(\boldsymbol{s}) + 0.4Z_{t-1}(\boldsymbol{s})\varepsilon_{t-1}(\boldsymbol{s}) + \varepsilon_t(\boldsymbol{s}).$$

We note that the nonlinear term $0.4Z_{t-1}(\boldsymbol{s})\varepsilon_{t-1}(\boldsymbol{s})$ induces sporadic bursts in the spatio-temporal process. The coefficients 0.5 and 0.4 are chosen to ensure that the process has a finite second moment (see Subba Rao and Gabr [1984] for details).

8.2.2 Discussion

The results for model S1 and S2 are given Table 1.

We first consider the stationary Gaussian model (S1). The results for all the tests are relatively good for both $\rho = 0.5$ and $\rho = 1$. However, for the average squared statistics (without variance adjustment) for $H = 10$ and $\rho = 1$ there are some inflations in the type I error. This is probably because without the variance adjustment the average squared statistics depend on the asymptotic result $W_{g,\widehat{V}^{-1/2}}(\omega_{jH}) \stackrel{\mathcal{P}}{\to} 1$ (see (37)) which depends on the range parameter ρ and the degree of non-Gaussianity. Model S1 is Gaussian, and it seems the error in this approximation seems only to mildly impact the case $H = 10$ and $\rho = 1$.

The results from the simulations for the stationary but non-Gaussian model (S2) are very different. The average covariance test results keep close to the nominal level (for both $\rho =$ 0.5 and 1) however there is a substantial inflation in the type I error (between 70-90%) for the average squared statistic without variance adjustment (for both $H = 10$ and $H = 20$).

This is likely due to the non-Gaussianity of the process which seems to greatly impact the rate that $W_{g,\widehat{V}^{-1/2}}(\omega_{jH}) \overset{\mathcal{P}}{\to} 1$. However, the variance adjusted average squared covariance test statistics appear to keep close to the nominal level for both $\rho = 0.5$ and 1 and $H = 10$ and 20. This demonstrates that $W_{g,\widehat{V}^{-1/2}}(\omega_{jH}) \overset{\mathcal{P}}{\to} 1$ is an *asymptotic* result and for finite samples it is important to estimate the variance.

In all cases, both removing and keeping the nugget term N_T give comparable results.

Our results in the simulation study demonstrate that both the average covariance and the variance adjusted average squared covariance test statistics perform well under the null, but caution needs to be taken when interpreting the results of the non-variance adjusted average squared covariance tests.

		Model S1				Model S2			
		ρ				ρ			
		0.5		1		0.5		1	
	n	100	500	100	500	100	500	100	500
	$\mathbf{T}_{1,g,\widehat{V}^{-1/2}}$	0.08 (0.08)	0.08 (0.07)	0.07 (0.07)	0.04 (0.04)	0.09 (0.09)	0.08 (0.09)	0.09 (0.08)	0.08 (0.07)
	$\mathbf{M}_{1,g,\widehat{V}^{-1/2}}$	0.04 (0.07)	0.05 (0.06)	0.05 (0.06)	0.02 (0.02)	0.06 (0.04)	0.05 (0.06)	0.07 (0.07)	0.06 (0.08)
H=20	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.04 (0.03)	0.01 (0.01)	0.06 (0.04)	0.03 (0.02)	0.45 (0.70)	0.86 (0.91)	0.62 (0.72)	0.94 (0.98)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.07 (0.08)	0.06 (0.05)	0.07 (0.08)	0.02 (0.03)	0.48 (0.67)	0.80 (0.88)	0.60 (0.70)	0.86 (0.88)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.05 (0.05)	0.02 (0.01)	0.07 (0.03)	0.02 (0.01)	0.06 (0.07)	0.05 (0.04)	0.05 (0.05)	0.06 (0.05)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.05 (0.06)	0.03 (0.03)	0.05 (0.05)	0.06 (0.05)	0.04 (0.07)	0.06 (0.07)	0.10 (0.09)	0.08 (0.08)
H=10	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.10 (0.10)	0.08 (0.07)	0.12 (0.12)	0.13 (0.14)	0.50 (0.71)	0.85 (0.90)	0.67 (0.78)	0.85 (0.88)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.08 (0.08)	0.09 (0.08)	0.11 (0.12)	0.16 (0.15)	0.42 (0.58)	0.65 (0.77)	0.53 (0.67)	0.79 (0.83)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.10 (0.08)	0.04 (0.04)	0.06 (0.04)	0.04 (0.04)	0.09 (0.10)	0.06 (0.06)	0.05 (0.06)	0.08 (0.10)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.05 (0.06)	0.03 (0.04)	0.06 (0.05)	0.05 (0.05)	0.08 (0.10)	0.09 (0.09)	0.10 (0.13)	0.11 (0.12)

Table 1: Empirical type I errors at 5% level based on different tests with $\lambda = 5$ for Gaussian and non-Gaussian stationary data with innovations coming from a Gaussian random field with exponential covariance functions. Rejection rate without removing N_T (see (19)) are in the parentheses.

8.2.3 Simulations under null using the Whittle spatial covariance

Up to this point all the simulations were conducted using an exponential spatial covariance. In this section our aim is to understand the behavior of the stationarity test for other spatial covariance functions. A popular spatial covariance commonly used in spatial statistics, is the Whittle correlation function (i.e., a Matérn correlation with smoothness parameter $\nu = 1$). Note that, the feature that distinguishes the exponential from Whittle is that around zero the exponential is peaked with no derivative whereas the Whittle is smooth. In this section we consider again the spatio-temporal models (S1 and S2) defined in Section 8.2.1, but generate the independent innovations $\varepsilon_t(\boldsymbol{s})$ with a Gaussian process with a spatial Whittle covariance i.e. $\text{cov}[\varepsilon_t(\boldsymbol{s}_1), \varepsilon_t(\boldsymbol{s}_2)] = (\|\boldsymbol{s}_2 - \boldsymbol{s}_2\|_2/\rho)\mathcal{K}_1(\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|_2/\rho)$, where $\mathcal{K}_1$ is the second kind modified Bessel function of order one. From the plots (Figure 2a and 2b) we see that the range parameter, ρ, in both exponential and Whittle do not match. For example, when $\rho = 0.5$ or $\rho = 1$ the Whittle correlation has much thicker tails than the exponential with the same range parameter.

Since our test is based on the DFTs being close to uncorrelated (and this property is lost when there is a lot of correlation in the original process compared with the size of the random field), the Whittle model is expected to give larger type I errors compared with the exponential model with the same range parameter. Therefore, in order to fairly compare the simulations using the exponential and Whittle spatial covariances, we adjust the range parameters for the Whittle correlations to have similar tail behavior as the exponential covariance. From a visual inspection we find that the Whittle covariance with range parameters $\rho = 0.37$ and $\rho = 0.72$ are the closest 'match' to the exponential correlations with range parameters $\rho = 0.5$ and $\rho = 1$, respectively (see, Figure 2a and 2b).

Based on the above setup the results are reported in Table 2. Comparing Tables 1 and 2 we observe that for the proportion of rejections the results are quite similar to what we noticed for exponential correlations with $\rho = 0.5$ and 1. This suggests that the test for stationarity is robust to different of types of stationary behavior.

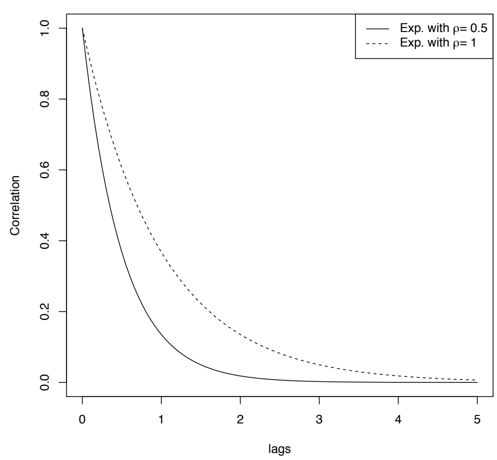


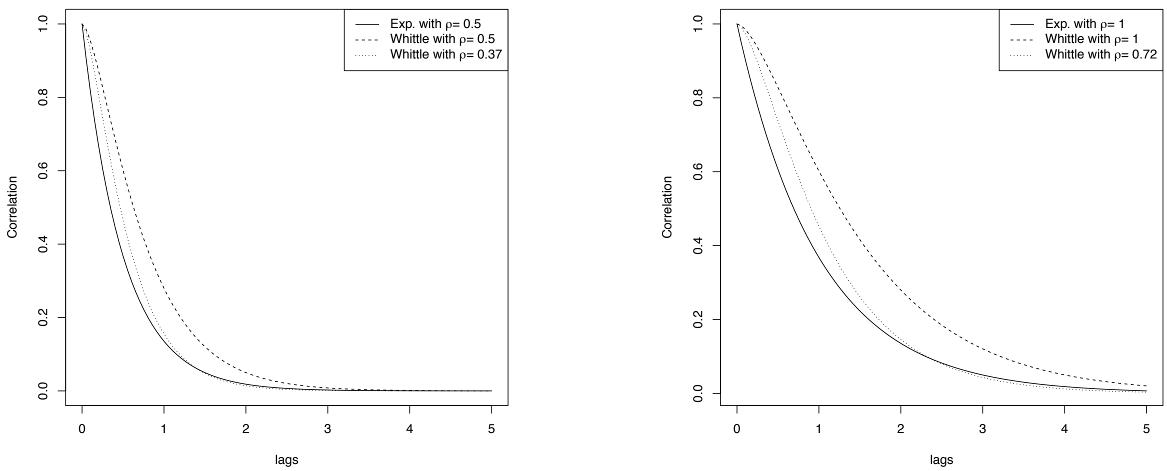
Figure 1: Plot of the Exponential correlation function with the range parameters $\rho = 0.5$ and $\rho = 1$.

8.3 Simulations under the alternative

8.3.1 Models

In order to induce spatial nonstationarity in the models (NS2) and (NS3) (defined below) we define the Gaussian innovations process $\{\eta_t(\boldsymbol{s}); \boldsymbol{s} \in [-\lambda/2, \lambda/2]^2\}$, which is independent over time with nonstationary covariance $\text{cov}[\eta_t(\boldsymbol{s}_1), \eta_t(\boldsymbol{s}_2)] = c_\lambda(\boldsymbol{s}_1, \boldsymbol{s}_2) = \kappa_0(\boldsymbol{s}_2 - \boldsymbol{s}_1; \boldsymbol{s}_1)$ where,

$$c_\lambda(\boldsymbol{s}_1, \boldsymbol{s}_2) = |\Sigma\left(\frac{\boldsymbol{s}_1}{\lambda}\right)|^{1/4}|\Sigma\left(\frac{\boldsymbol{s}_2}{\lambda}\right)|^{1/4}\left|\frac{\Sigma(\frac{\boldsymbol{s}_1}{\lambda}) + \Sigma(\frac{\boldsymbol{s}_2}{\lambda})}{2}\right|^{-1/2}\exp[-\sqrt{Q_\lambda(\boldsymbol{s}_1, \boldsymbol{s}_2)}],$$



(a) (b)

Figure 2: Plots of exponential and Whittle correlations with different range parameters.

		Model S1				Model S2			
		ρ				ρ			
		0.37		0.72		0.37		0.72	
	n	100	500	100	500	100	500	100	500
	$\mathbf{T}_{1,g,\widehat{V}^{-1/2}}$	0.07 (0.06)	0.05 (0.05)	0.08 (0.08)	0.08 (0.08)	0.09 (0.09)	0.08 (0.08)	0.09 (0.08)	0.07 (0.08)
	$\mathbf{M}_{1,g,\widehat{V}^{-1/2}}$	0.06 (0.07)	0.06 (0.06)	0.08 (0.06)	0.04 (0.04)	0.07 (0.08)	0.06 (0.07)	0.09 (0.09)	0.06 (0.08)
H=20	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.05 (0.04)	0.02 (0.02)	0.04 (0.04)	0.05 (0.04)	0.55 (0.73)	0.97 (0.98)	0.76 (0.82)	0.95 (0.95)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.06 (0.05)	0.05 (0.05)	0.07 (0.06)	0.06 (0.06)	0.50 (0.68)	0.88 (0.90)	0.72 (0.77)	0.92 (0.92)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.04 (0.04)	0.01 (0.03)	0.05 (0.04)	0.04 (0.04)	0.06 (0.05)	0.03 (0.04)	0.06 (0.07)	0.04 (0.04)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.06 (0.05)	0.04 (0.03)	0.06 (0.07)	0.04 (0.04)	0.06 (0.05)	0.05 (0.07)	0.09 (0.11)	0.08 (0.09)
H=10	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.12 (0.12)	0.09 (0.09)	0.13 (0.13)	0.15 (0.15)	0.61 (0.74)	0.85 (0.96)	0.75 (0.82)	0.91 (0.92)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.09 (0.09)	0.10 (0.10)	0.13 (0.13)	0.14 (0.12)	0.46 (0.62)	0.82 (0.86)	0.66 (0.72)	0.84 (0.88)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.08 (0.08)	0.03 (0.02)	0.06 (0.04)	0.06 (0.06)	0.09 (0.10)	0.06 (0.07)	0.05 (0.06)	0.06 (0.06)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.07 (0.07)	0.04 (0.03)	0.06 (0.05)	0.06 (0.06)	0.08 (0.10)	0.09 (0.09)	0.09 (0.12)	0.07 (0.10)

Table 2: Empirical type I errors at 5% level based on different tests with $\lambda = 5$ for Gaussian and non-Gaussian stationary data with innovations coming from a Gaussian random field with Whittle covariance functions. Rejection rate without removing N_T (see (19)) are in the parentheses.

$|\cdot|$ denotes the determinant of a matrix, $Q_\lambda(\boldsymbol{s}_1, \boldsymbol{s}_2) = 2(\boldsymbol{s}_1 - \boldsymbol{s}_2)'[\Sigma(\frac{\boldsymbol{s}_1}{\lambda}) + \Sigma(\frac{\boldsymbol{s}_2}{\lambda})]^{-1}(\boldsymbol{s}_1 - \boldsymbol{s}_2)$ and $\Sigma(\frac{\boldsymbol{s}}{\lambda}) = \Gamma(\frac{\boldsymbol{s}}{\lambda})\Lambda\Gamma(\frac{\boldsymbol{s}}{\lambda})'$, where

$$\Gamma\left(\frac{\boldsymbol{s}}{\lambda}\right) = \begin{bmatrix} \gamma_1(\boldsymbol{s}/\lambda) & -\gamma_2(\boldsymbol{s}/\lambda) \\ \gamma_2(\boldsymbol{s}/\lambda) & \gamma_1(\boldsymbol{s}/\lambda) \end{bmatrix}, \ \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

with $\gamma_1(\boldsymbol{s}/\lambda) = \log(s_x/\lambda + 0.75)$, $\gamma_2(\boldsymbol{s}/\lambda) = (s_x/\lambda)^2 + (s_y/\lambda)^2$, and $\boldsymbol{s} = (s_x, s_y)'$ (see Paciorek and Schervish [2006] and Jun and Genton [2012] for the details on this process). Note that the variance of this process is constant over the spatial random field and it is simply the correlation structure that varies over space.

(NS1) *Temporally nonstationary but spatially stationary Gaussian random field:* $Z_t(\boldsymbol{s}) = 0.5Z_{t-1}(\boldsymbol{s}) + \left(1.3 + \sin\left(\frac{2\pi t}{400}\right)\right)\varepsilon_t(\boldsymbol{s})$, where $\{\varepsilon_t(\boldsymbol{s})\}$ is defined in Section 8.2.1. We use $\rho = 0.5$, $\rho = 1$ and $\lambda = 5$.

(NS2) *Temporally stationary but spatially nonstationary Gaussian random field:* The spatio-temporal process is defined with an AR(1) model $Z_t(\boldsymbol{s}) = 0.5Z_{t-1}(\boldsymbol{s}) + \eta_t(\boldsymbol{s})$. Following a similar set-up as in Bandyopadhyay and Subba Rao [2016] we use $\lambda = 20$. This process has a constant variance over space and time.

(NS3) *Both temporally and spatially nonstationary Gaussian random field:* The spatio-temporal process is defined using an AR(1) model with time-dependent innovations

$$Z_t(\boldsymbol{s}) = 0.5Z_{t-1}(\boldsymbol{s}) + \left(1.3 + \sin\left(\frac{2\pi t}{400}\right)\right)\eta_t(\boldsymbol{s}).$$

For the simulations we use $\lambda = 20$.

8.3.2 Discussion

The empirical powers based on Models NS1 - NS3 are given in Table 3-6.

First we consider Model NS1, which is temporarily nonstationary, but stationary over space. The results of the general spatio-temporal test using the test set $\mathcal{P} = \{(1,0),(1,1),(0,1),(-1,1)\} \times \{1,2\}$ and orthogonal estimates set $\mathcal{P}' = \{(1,0),(1,1),(0,1),(-1,1)\} \times \{4,5\}$ (described in Section 4) are given in Table 3. Before discussing the results we note that over the test set $\mathcal{P}$ the Fourier transforms are near uncorrelated. However, the temporal nonstationarity means that the orthogonal estimators $\widehat{A}_{g,h}(\boldsymbol{r}_1, r_2)$ and $\widehat{B}_{g,h;H}(\omega_{jH}; \boldsymbol{r}_1, r_2)$ for $(\boldsymbol{r}_1, r_2) \in \mathcal{P}'$ do not necessarily share the same variance. Furthermore, there is correlation between the terms. These conflicting behaves (decorrelation of DFTs but inability to capture the true variance) helps explain why the power in the overall test varies between 27%-80% in the case $\rho = 0.5$ and 21% - 80% in the case $\rho = 1$ (excluding the non-variance adjusted tests). The results of the one-way temporal stationary and one-way spatial stationary tests (described in Section 5) are given in Table 4. The power in the one-way temporal tests are close to 100% for all the test statistics (as we would expect since the process is temporally nonstationary) for both $\rho = 0.5$ and $\rho = 1$. The power for the one-way spatial tests drops considerably (as expected because $NS1$ is spatially stationary) for the average covariance test and variance adjusted average squared covariance test. In the case of the variance adjusted average squared tests the proportion of rejection is least in the case $\rho = 0.5$ and $H = 10$.

Next we consider Model NS2, which is temporarily stationary, but spatially nonstationary. The results are reported in Table 5. In this case the general spatio-temporal test using the test set $\mathcal{P} = \{(1,0),(1,1),(0,1),(-1,1)\} \times \{1,2\}$ and orthogonal estimates set $\mathcal{P}' = \{(1,0),(1,1),(0,1),(-1,1)\} \times \{4,5\}$ gives very little power. As we would expect, in the one-way test for temporal stationarity the proportion of rejections is close to the nominal level (with the exception of the variance adjusted test average squared test with $H = 20$ when the proportion of rejection is about 12%). However, the test does seem to have some power in the one-way test for spatial stationarity. In the case that $n = 500$ all the tests (excluding the non-variance adjusted tests) have power between 8-21%. This level of power is not high but it is higher than the case $n = 100$. The overall low power is because the number of observations is relatively sparse on the random field ($n = 500$ and $\lambda = 20$). Therefore most of the observations are unlikely to be highly correlated and thus contains very little information about the nonstationary structure (recall the variance of the spatio-temporal process is constant). It is likely if a larger n were used in the simulations, the power would increase (compare with the simulations in Bandyopadhyay and Subba Rao [2016]).

Lastly, we consider Model NS3, which is both temporal and spatial nonstationarity. The results are presented in Table 6. For the general spatio-temporal tests we get higher powers than for Model NS1 across all the tests. The power increases to 100% for the one-way temporal stationary test. For the one-way spatial stationarity tests the power is more than for the same tests using model NS2.

We mention that for all the models (NS1-NS3) the power for the average squared covariance test without variance adjustment is very high. However, we have to be cautious about interpreting the result of these tests as the simulations under the null of stationarity show that the these test statistics are unable to keep the nominal level when the process is not Gaussian.

Comparing the rejection rates with and without the nugget term removed (the values outside and insides the parentheses), we observe that for models NS1 and NS2 the rejection rates with and without the nugget term are about the same. However, for NS3 the power is slightly more after removing the nugget term.

		Model NS1: Overall Power			
		ρ			
		0.5		1	
	n	100	500	100	500
	$\mathbf{T}_{1,g,\widehat{V}^{-1/2}}$	0.73 (0.80)	0.60 (0.59)	0.76 (0.74)	0.57 (0.49)
	$\mathbf{M}_{1,g,\widehat{V}^{-1/2}}$	0.74 (0.78)	0.61 (0.59)	0.80 (0.75)	0.59 (0.51)
H=20	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.99 (0.99)	0.99 (0.99)	0.99 (1.00)	0.97 (0.96)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.97 (0.99)	0.99 (0.99)	0.98 (0.98)	0.93 (0.92)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.44 (0.56)	0.33 (0.27)	0.45 (0.45)	0.22 (0.21)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.51 (0.64)	0.47 (0.45)	0.60 (0.57)	0.34 (0.27)
H=10	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	1.00 (1.00)	1.00 (0.99)	0.99 (1.00)	1.00 (1.00)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.98 (0.99)	0.99 (0.98)	0.99 (0.98)	0.98 (0.97)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.53 (0.56)	0.39 (0.37)	0.55 (0.50)	0.30 (0.27)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.52 (0.56)	0.46 (0.44)	0.52 (0.48)	0.33 (0.28)

Table 3: Overall empirical power at 5% level based on different tests with $\lambda = 5$ for nonstationary data generated from the model NS1 with innovations coming from a Gaussian random field with exponential covariance functions. Rejection rate without removing N_T (see (19)) are in the parentheses.

		Model NS1							
		Temporal Power				Spatial Power			
		ρ				ρ			
		0.5		1		0.5		1	
	n	100	500	100	500	100	500	100	500
	$\mathbf{T}_{1,g,\widehat{V}^{-1/2}}$	0.97 (0.99)	1.00 (1.00)	0.97 (0.99)	1.00 (1.00)	0.01 (0.01)	0.01 (0.01)	0.04 (0.03)	0.04 (0.04)
	$\mathbf{M}_{1,g,\widehat{V}^{-1/2}}$	0.99 (1.00)	1.00 (1.00)	0.99 (0.99)	1.00 (1.00)	0.01 (0.01)	0.02 (0.01)	0.02 (0.02)	0.03 (0.01)
H=20	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.99 (0.99)	0.99 (0.99)	1.00 (1.00)	1.00 (1.00)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.99 (0.98)	0.99 (0.99)	1.00 (1.00)	1.00 (1.00)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.05 (0.04)	0.10 (0.08)	0.15 (0.14)	0.28 (0.31)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.15 (0.15)	0.18 (0.16)	0.23 (0.24)	0.31 (0.32)
H=10	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.99 (0.99)	0.99 (0.99)	1.00 (1.00)	1.00 (1.00)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.99 (0.99)	0.99 (0.99)	1.00 (1.00)	1.00 (1.00)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.02 (0.02)	0.06 (0.05)	0.10 (0.11)	0.26 (0.28)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.06 (0.06)	0.09 (0.07)	0.14 (0.14)	0.24 (0.26)

Table 4: One-way empirical powers at 5% level based on different tests with $\lambda = 5$ for nonstationary data generated from the model NS1 with innovations coming from a Gaussian random field with exponential covariance functions. Rejection rate without removing N_T (see (19)) are in the parentheses.

		Model NS2					
		Overall Power		Temporal Power		Spatial Power	
	n	100	500	100	500	100	500
	$\mathbf{T}_{1,g,\widehat{V}^{-1/2}}$	0.09 (0.11)	0.11 (0.11)	0.04 (0.05)	0.05 (0.06)	0.06 (0.07)	0.17 (0.17)
	$\mathbf{M}_{1,g,\widehat{V}^{-1/2}}$	0.08 (0.08)	0.10 (0.09)	0.03 (0.03)	0.05 (0.06)	0.03 (0.02)	0.15 (0.15)
H=20	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.04 (0.06)	0.06 (0.06)	0.04 (0.05)	0.05 (0.06)	0.24 (0.26)	0.28 (0.31)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.07 (0.09)	0.07 (0.07)	0.08 (0.08)	0.03 (0.04)	0.12 (0.15)	0.21 (0.25)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.07 (0.06)	0.05 (0.04)	0.11 (0.12)	0.12 (0.12)	0.05 (0.06)	0.18 (0.19)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.08 (0.06)	0.08 (0.09)	0.11 (0.12)	0.13 (0.12)	0.07 (0.10)	0.21 (0.20)
H=10	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.15 (0.18)	0.15 (0.15)	0.06 (0.07)	0.06 (0.07)	0.38 (0.47)	0.56 (0.59)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.12 (0.10)	0.13 (0.15)	0.05 (0.06)	0.06 (0.06)	0.32 (0.34)	0.48 (0.50)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.08 (0.10)	0.06 (0.05)	0.05 (0.04)	0.04 (0.04)	0.01 (0.02)	0.09 (0.10)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.13 (0.14)	0.10 (0.10)	0.06 (0.03)	0.04 (0.05)	0.01 (0.01)	0.08 (0.09)

Table 5: Empirical powers at 5% level based on different tests with $\lambda = 20$ for nonstationary data generated from the model NS2. Rejection rate without removing N_T (see (19)) are in the parentheses.

		Model NS3					
		Overall Power		Temporal Power		Spatial Power	
	n	100	500	100	500	100	500
	$\mathbf{T}_{1,g,\widehat{V}^{-1/2}}$	0.83 (0.80)	0.98 (0.92)	0.92 (0.99)	1.00 (1.00)	0.11 (0.07)	0.33 (0.19)
	$\mathbf{M}_{1,g,\widehat{V}^{-1/2}}$	0.92 (0.88)	0.99 (0.97)	0.95 (1.00)	1.00 (1.00)	0.18 (0.08)	0.54 (0.25)
H=20	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	0.99 (0.99)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	0.99 (1.00)	1.00 (1.00)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.99 (0.98)	1.00 (0.99)	1.00 (1.00)	1.00 (1.00)	0.99 (1.00)	1.00 (1.00)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.65 (0.66)	0.85 (0.77)	1.00 (1.00)	1.00 (1.00)	0.34 (0.22)	0.74 (0.50)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.82 (0.80)	0.98 (0.87)	1.00 (1.00)	1.00 (1.00)	0.52 (0.34)	0.90 (0.69)
H=10	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},1}$	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},1}$	0.99 (0.99)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)	1.00 (1.00)
	$\mathbf{T}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.65 (0.63)	0.81 (0.77)	1.00 (1.00)	1.00 (1.00)	0.13 (0.08)	0.45 (0.35)
	$\mathbf{M}_{2,g,\widehat{V}^{-1/2},\widehat{W}}$	0.79 (0.70)	0.94 (0.84)	1.00 (1.00)	1.00 (1.00)	0.32 (0.19)	0.79 (0.58)

Table 6: Empirical powers at 5% level based on different tests with $\lambda = 20$ for nonstationary data generated from the model NS3. Rejection rate without removing N_T (see (19)) are in the parentheses.

A Proofs

A.1 Proof of Lemma 3.1

To prove the result we start by expanding $\text{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}, \omega_{k_2+r_2})\right]$.

$$\begin{aligned}
&\text{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}, \omega_{k_2+r_2})\right] \\
&= \frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1} e^{-ih\omega_{k_2}} \frac{1}{T}\sum_{t=1-\min(0,h)}^{T-\max(0,h)} \text{cov}[J_t(\boldsymbol{\Omega}_{k_1}), J_{t+h}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})]e^{-it\omega_{r_2}} \\
&= M + R, \qquad (49)
\end{aligned}$$

where M is the main term

$$M = \frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1} e^{-ih\omega_{k_2}} \frac{1}{T}\sum_{t=1}^{T} \text{cov}[J_t(\boldsymbol{\Omega}_{k_1}), J_{t+h}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})]e^{-it\omega_{r_2}},$$

and R is the remainder

$$\begin{aligned}
R &= \frac{1}{2\pi}\sum_{h=0}^{T-1} e^{-ih\omega_{k_2}} \frac{1}{T}\sum_{t=T-h+1}^{T} \text{cov}[J_t(\boldsymbol{\Omega}_{k_1}), J_{t+h}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})]e^{-it\omega_{r_2}} \\
&\quad + \frac{1}{2\pi}\sum_{h=-(T-1)}^{-1} e^{-ih\omega_{k_2}} \frac{1}{T}\sum_{t=1}^{|h|} \text{cov}[J_t(\boldsymbol{\Omega}_{k_1}), J_{t+h}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})]e^{-it\omega_{r_2}}.
\end{aligned}$$

The expansions above are valid in the general case. Below we obtain expressions for M (the main term) and bounds for R in the case that the spatio-temporal process is stationary and nonstationary.

- **Spatially stationary**

  By using the same proof used to prove Theorem 2.1(i), Bandyopadhyay and Subba Rao [2016], and the rescaling devise over time, under spatial stationary we have, for $\boldsymbol{r}_1 \neq \boldsymbol{0}$,

$$\begin{aligned}
&\text{cov}[J_t(\boldsymbol{\Omega}_{\boldsymbol{k}_1}), J_{t+h}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})] \\
&= \underbrace{\int_{[-\lambda/2,\lambda/2]^d} e^{-i\boldsymbol{\Omega}_{\boldsymbol{r}}'\boldsymbol{s}_2} \int_{[-\lambda/2-\boldsymbol{s}_1,-\lambda/2]^d} \kappa_{h;\frac{t}{T}}(\boldsymbol{s}_1)e^{i\boldsymbol{\Omega}_{\boldsymbol{k}_1}'\boldsymbol{s}_1}d\boldsymbol{s}_1 d\boldsymbol{s}_2}_{O(\frac{\rho_h}{\lambda^{d-b}})} \\
&+ \underbrace{\int_{[-\lambda/2,\lambda/2]^d} e^{-i\boldsymbol{\Omega}_{\boldsymbol{r}}'\boldsymbol{s}_2} \int_{[\lambda/2,\lambda/2+\boldsymbol{s}_1]^d} \kappa_{h;\frac{t}{T}}(\boldsymbol{s}_1)e^{i\boldsymbol{\Omega}_{\boldsymbol{k}_1}'\boldsymbol{s}_1}d\boldsymbol{s}_1 d\boldsymbol{s}_2}_{O(\frac{\rho_h}{\lambda^{d-b}})} + O\left(\frac{\rho_h}{T\lambda^{d-b}} I_{\text{Time=NS}}\right),
\end{aligned}$$

and for $\boldsymbol{r}_1 = \boldsymbol{0}$,

$$\begin{aligned}&\mathrm{cov}[J_t(\boldsymbol{\Omega}_{\boldsymbol{k}_1}), J_{t+h}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})]\\&= \frac{c_{t,t+h}}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^d}\kappa_{h;\frac{t}{T}}(\boldsymbol{v})\exp(-i\boldsymbol{v}'\boldsymbol{\Omega}_{\boldsymbol{k}_1})d\boldsymbol{v} + \frac{\lambda^d n_{t,t+h}}{n_t n_{t+h}}\kappa_{h,\frac{t}{T}}(0) + O\left(\frac{\rho_h I_{\text{Time=NS}}}{T}\right),\end{aligned}$$

where, $b = b(\boldsymbol{r}_1)$ is the number of zeros in $\boldsymbol{r}_1$, $c_{t,t+h} = (n_t n_{t+h} - n_{t,t+h})/n_t n_{t+h}$, $n_{t,t+h} = |\{\boldsymbol{s}_{t,j}\}_{j=1}^{n_t} \cap \{\boldsymbol{s}_{t+h,j}\}_{j=1}^{n_{t+h}}|$ and $I_{\text{Time=NS}}$ denotes the indicator variable for temporal nonstationarity. Note that we use the notation $[-\lambda/2 - \boldsymbol{s}_1, -\lambda/2]^d = [-\lambda/2 - s_{11}] \times \ldots \times [-\lambda/2 - s_{1d}]$. Substituting the above into the remainder R we see that $|R| = O([T^{-1} + \lambda^d/n]I(\boldsymbol{r}_1 = 0) + \frac{1}{\lambda^{d-b}T}I(\boldsymbol{r}_1 \neq 0))$. Now we derive expression for M for the temporally stationary and nonstationary separately.

(a) **Temporally stationary** (i.e., $\kappa_{h,\frac{t}{T}}(\boldsymbol{v}) = \kappa_h(\boldsymbol{v})$) First we look at the case $\boldsymbol{r}_1 \neq \boldsymbol{0}$. In the case that $\boldsymbol{r}_1 \neq \boldsymbol{0}$ and $r_2 \neq 0$, we take the summand $\sum_{t=1}^{T} e^{-it\omega_{r_2}}$ in M separate of κ_h giving $M = 0$. Therefore, $\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}, \omega_{k_2+r_2})\right] = O(\lambda^{-(d-b)}T^{-1})$. In the case that $\boldsymbol{r}_1 \neq \boldsymbol{0}$ but $r_2 = 0$, we get $M = O(\lambda^{-(b-d)})$, and thus $\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}, \omega_{k_2+r_2})\right] = O(\lambda^{-(d-b)})$.

Now we consider the case $\boldsymbol{r}_1 = \boldsymbol{0}$. In the case that $\boldsymbol{r}_1 = \boldsymbol{0}$ but $r_2 \neq 0$, we use Assumption 3.1(ii), where $c_1 n \leq n_t \leq c_2 n$, which implies that $|c_{t,t+h} - 1| \leq \frac{c_2}{c_1 n}$ and immediately gives $M = O(T^{-1} + \lambda^d/n)$ and $\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2+r_2})\right] = O(\lambda^d/n + T^{-1})$. On the other hand, when $\boldsymbol{r}_1 = \boldsymbol{0}$ *and* $r_2 = 0$ we have $M = f(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}) + O(T^{-1} + \lambda^{-1} + \lambda^d/n)$, which immediately leads us to $\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2})\right] = f(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}) + O(T^{-1} + \lambda^{-1} + \lambda^d/n)$.

(b) **Temporally nonstationary** Again it is immediately clear that when $\boldsymbol{r}_1 \neq \boldsymbol{0}$ ($r_2 \in \mathbb{Z}$) we have $M = O\left(\lambda^{-(d-b)}\right)$, which gives $\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1}, \omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}, \omega_{k_2+r_2})\right] = O(\lambda^{-(d-b)} + T^{-1})$. However, when $\boldsymbol{r}_1 = \boldsymbol{0}$ ($r_2 \in \mathbb{Z}$) (and using Assumption 3.1(ii)) it is clear that

$$M = \frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1} e^{-ih\omega_{k_2}} \frac{1}{T}\sum_{t=1}^{T}\frac{1}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^d}\kappa_{h;\frac{t}{T}}(\boldsymbol{v})\exp(-i\boldsymbol{v}'\boldsymbol{\Omega}_{\boldsymbol{k}_1})d\boldsymbol{v} + O\left(\frac{\lambda^d}{n} + \frac{1}{T}\right),$$

which gives the desired result.

- **Spatially nonstationary** If the spatio-temporal process is spatially nonstationary, using the same proof to prove Theorem 2.1(ii), Bandyopadhyay and Subba Rao [2016] and the rescaling devise over time and space we have,

$$
\begin{aligned}
&\mathrm{cov}[J_t(\boldsymbol{\Omega}_{\boldsymbol{k}_1}), J_{t+h}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})] \\
&= \frac{c_{t,t+h}}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^{2d}} \kappa_{h;\frac{t}{T}}\left(\boldsymbol{v};\frac{\boldsymbol{s}}{\lambda}\right)\exp(-i\boldsymbol{v}'\boldsymbol{\Omega}_{\boldsymbol{k}_1})\exp(-i\boldsymbol{s}'\boldsymbol{\Omega}_{\boldsymbol{r}_1})d\boldsymbol{v}d\boldsymbol{s} \\
&+ \underbrace{\int_{[-\lambda/2,\lambda/2]^d} e^{-i\boldsymbol{\Omega}_{\boldsymbol{r}}'\boldsymbol{s}}\int_{[-\lambda/2-\boldsymbol{s},-\lambda/2]^d}\kappa_{h;\frac{t}{T}}\left(\boldsymbol{v};\frac{\boldsymbol{s}}{\lambda}\right)e^{i\boldsymbol{\Omega}_{\boldsymbol{k}_1}'\boldsymbol{v}}d\boldsymbol{v}d\boldsymbol{s}}_{O(\rho_h/\lambda)} \\
&+ \underbrace{\int_{[-\lambda/2,\lambda/2]^d} e^{-i\boldsymbol{\Omega}_{\boldsymbol{r}}'\boldsymbol{s}}\int_{[\lambda/2,\lambda/2+\boldsymbol{s}]^d}\kappa_{h;\frac{t}{T}}\left(\boldsymbol{v};\frac{\boldsymbol{s}}{\lambda}\right)e^{i\boldsymbol{\Omega}_{\boldsymbol{k}_1}'\boldsymbol{v}}d\boldsymbol{v}d\boldsymbol{s}}_{O(\rho_h/\lambda)} \\
&+\frac{n_{t,t+h}}{n_t n_{t+h}}\int_{[-\lambda/2,\lambda/2]^d}\kappa_{h;\frac{t}{T}}\left(0;\frac{\boldsymbol{s}}{\lambda}\right)\exp(-i\boldsymbol{s}'\boldsymbol{\Omega}_{\boldsymbol{r}_1})d\boldsymbol{s} + O\left(\frac{\rho_h I_{\mathrm{Time=NS}}}{T}\right).
\end{aligned}
$$

Using the above result it is straightforward to show that $R = O([1+\lambda^d/n]T^{-1})$.

(a) **Temporally stationary** (i.e., $\kappa_{h,\frac{t}{T}}(\boldsymbol{v},\boldsymbol{s}) = \kappa_h(\boldsymbol{v},\boldsymbol{s})$). Since the process is spatially nonstationary, we consider $\boldsymbol{r}_1 = \boldsymbol{0}$ and $\boldsymbol{r}_1 \neq 0$ together. In the case that $r_2 \neq 0$ $\sum_{t=1}^{T} e^{-itr_2}$ is separate of κ_h, thus $M = 0$ and $\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1},\omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_1},\omega_{k_2})\right] = O(T^{-1})$.

If $r_2 = 0$ we have,

$$
M = \frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1} e^{-ih\omega_{k_2}}\frac{1}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^d}\kappa_h\left(\boldsymbol{v};\frac{\boldsymbol{s}}{\lambda}\right)\exp(-i\boldsymbol{v}'\boldsymbol{\Omega}_{\boldsymbol{k}_1})\exp(-i\boldsymbol{s}'\boldsymbol{\Omega}_{\boldsymbol{r}_1})d\boldsymbol{v}d\boldsymbol{s}+O\left(\frac{1}{\lambda}\right),
$$

which immediately leads to the desired result.

(b) **Temporally nonstationary** In this case using Assumption 3.1(ii) we have,

$$
\begin{aligned}
M &= \frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1} e^{-ih\omega_{k_2}}\frac{1}{T}\sum_{t=1}^{T} e^{-it\omega_{r_2}} \\
&\quad\times\frac{c_{t,t+h}}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^{2d}}\kappa_{h;\frac{t}{T}}\left(\boldsymbol{v};\frac{\boldsymbol{s}}{\lambda}\right)\exp(-i\boldsymbol{v}'\boldsymbol{\Omega}_{\boldsymbol{k}_1})\exp(-i\boldsymbol{s}'\boldsymbol{\Omega}_{\boldsymbol{r}_1})d\boldsymbol{v}d\boldsymbol{s} + O\left(\frac{\lambda^d}{n}+\frac{1}{\lambda}\right),
\end{aligned}
$$

thus leading to the desired result.

A.2 Proof of results for stationary spatio-temporal processes

PROOF of Lemma 4.1 The proof of this lemma is identical to the proof of Lemma 3.1 in Bandyopadhyay and Subba Rao [2016] and hence omitted. □

To prove the remainder of the results in Section 4 we use the following notation

$$f_h(\boldsymbol{\Omega}) = \int_{\mathbb{R}^d} \kappa_h(\boldsymbol{s}) \exp(-i\boldsymbol{\Omega}'\boldsymbol{s}) d\boldsymbol{s},$$

$$f(\boldsymbol{\Omega},\omega) = \sum_{h\in\mathbb{Z}} \exp(-ih\omega) \int_{\mathbb{R}^d} \kappa_h(\boldsymbol{s}) \exp(-i\boldsymbol{\Omega}'\boldsymbol{s}) d\boldsymbol{s},$$

and

$$f_{h_1,h_2,h_3}(\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2,\boldsymbol{\Omega}_3) = \int_{\mathbb{R}^{3d}} \kappa_{h_1,h_2,h_3}(\boldsymbol{s}_1,\boldsymbol{s}_2,\boldsymbol{s}_3) \exp(-i(\boldsymbol{s}_1'\boldsymbol{\Omega}_1 + \boldsymbol{s}_2'\boldsymbol{\Omega}_2 + \boldsymbol{s}_3'\boldsymbol{\Omega}_3)) d\boldsymbol{s}_1 d\boldsymbol{s}_2 d\boldsymbol{s}_3.$$

Note that in this section we do not prove any central limit theorems. However, we conjecture that by combining Bandyopadhyay et al. [2015], which give a CLT for mixing spatial processes and the CLT for quadratic forms of a time series (see, for example, Hsing and Wu [2004], Leucht [2012], Lee and Subba Rao [2015]) asymptotic normality of spatio-temporal quadratic forms can be proved.

Having established an expression for the mean of $\widehat{a}_g(\cdot)$ under stationarity, the main focus is obtaining expressions for the variance and covariance of $\widehat{a}_g(\cdot)$ and the corresponding test statistics. To do this we define the related quantity $\widetilde{a}_g(\cdot)$ such that

$$\widetilde{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2) = \widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2) + N_T,$$

where,

$$N_T = \frac{1}{2\pi T}\sum_{t,\tau=1}^{T} e^{it\omega_{k_2} - i\tau\omega_{k_2+r_2}} \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1=-\boldsymbol{a}}^{\boldsymbol{a}} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1}) \frac{1}{n_t n_\tau}\sum_{j=1}^{n} \delta_{t,j}\delta_{\tau,j} Z_t(\boldsymbol{s}_j) Z_\tau(\boldsymbol{s}_j) e^{-i\boldsymbol{s}_j\boldsymbol{\Omega}_{\boldsymbol{r}_1}}.$$

More precisely, we have,

$$\begin{aligned}
&\widetilde{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2) \\
=\ & \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1=-\boldsymbol{a}}^{\boldsymbol{a}} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1}) J(\boldsymbol{\Omega}_{k_1},\omega_{k_2})\overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2})} \\
=\ & \frac{1}{2\pi T}\sum_{t,\tau=1}^{T} e^{it\omega_{k_2} - i\tau\omega_{k_2+r_2}} \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1=-\boldsymbol{a}}^{\boldsymbol{a}} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1}) J_t(\boldsymbol{\Omega}_{k_1})\overline{J_\tau(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})} \\
=\ & \frac{1}{2\pi T}\sum_{t,\tau=1}^{T} e^{it\omega_{k_2} - i\tau\omega_{k_2+r_2}} \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1=-\boldsymbol{a}}^{\boldsymbol{a}} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1}) \frac{1}{n_t n_\tau}\sum_{j_1,j_2=1}^{n} \delta_{t,j_1}\delta_{\tau,j_2} Z_t(\boldsymbol{s}_{j_1}) Z_\tau(\boldsymbol{s}_{j_2}) \\
&\quad \times e^{i\boldsymbol{s}_{j_1}\boldsymbol{\Omega}_{\boldsymbol{k}_1} - i\boldsymbol{s}_{j_2}\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}},
\end{aligned}$$

where the second equation follows by expanding $J(\mathbf{\Omega}_{k_1},\omega_{k_2})$. To understand the role N_T plays, consider the expectation of N_T for the case $\boldsymbol{r}_1=\mathbf{0}$ and $r_2=0$; not a case included in the text, but useful in understanding its role. Taking expectation of N_T (under stationarity) we have

$$\begin{aligned}
\mathrm{E}[N_T] &= \frac{1}{2\pi T}\sum_{\boldsymbol{k}=-\boldsymbol{a}}^{\boldsymbol{a}} g(\mathbf{\Omega}_{\boldsymbol{k}_1})\sum_{t,\tau=1}^{T}\exp(-i(\tau-t)\omega_{k_2})\frac{n_{t,\tau}}{n_t n_\tau} \\
&\approx \frac{\lambda^d}{n}\int_{2\pi[-a/\lambda,a/\lambda]^d} g(\mathbf{\Omega})d\mathbf{\Omega}\times\frac{1}{2\pi}\sum_{h\in\mathbb{Z}}\exp(-ih\omega_{k_2})\kappa_h(0)=O\left(\frac{a^d}{n}\right).
\end{aligned}$$

In the case that we constrain the frequency grid $\{\mathbf{\Omega}_{\boldsymbol{k}};\boldsymbol{k}=(k_1,\ldots,k_d),-a\le k_j\le a\}$ to be bounded, i.e., $a/\lambda\to c<\infty$ as $a,\lambda\to\infty$, then it is clear that $\mathrm{E}[N_T]=O(\lambda^d/n)=o(1)$. Furthermore, using similar arguments it can be shown that the variance of N_T is asymptotically negligible and $\lambda^d\mathrm{var}[\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)]=\lambda^d\mathrm{var}[\widetilde{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)]+o(1)$ when the frequency grid is bounded. On the other hand, if the frequency grid is not bounded and $a/\lambda\to\infty$ as $\lambda\to\infty$ then we can show that for $\boldsymbol{r}_1=\mathbf{0}$ and $r_2=0$ we have $\mathrm{E}[N_T]=O(a^d/n)$ and for general $\boldsymbol{r}_1$ and r_2 $\lambda^d\mathrm{var}[N_T]=(a^{2d}/n^2)$. Therefore, if the frequency grid is not bounded, $\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)$ and $\widetilde{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)$ are not asymptotically equivalent. However, $\widetilde{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)$ does play an important role in understanding the covariance of $\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)$, and we come back to this later on.

Returning to $\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)$, we see from the definition of $\widehat{a}_g(\cdot)$ that in order to obtain the covariance of $\widehat{a}_g(\cdot)$ we require the expansion

$$\begin{aligned}
&\lambda^d\mathrm{cov}\Bigg[\frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1=-a}^{a} g(\mathbf{\Omega}_{\boldsymbol{k}_1})\frac{1}{n_{t_1}n_{t_2}}\sum_{j_1\neq j_2}\delta_{t_1,j_1}\delta_{t_2,j_2}Z_{t_1}(\boldsymbol{s}_{j_1})Z_{t_2}(\boldsymbol{s}_{j_2})e^{i\boldsymbol{s}_{j_1}\mathbf{\Omega}_{\boldsymbol{k}_1}-i\boldsymbol{s}_{j_2}\mathbf{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}}, \\
&\quad\frac{1}{\lambda^d}\sum_{\boldsymbol{k}_3=-a}^{a} g(\mathbf{\Omega}_{\boldsymbol{k}_3})\frac{1}{n_{t_3}n_{t_4}}\sum_{j_3\neq j_4}\delta_{t_3,j_3}\delta_{t_4,j_4}Z_{t_4}(\boldsymbol{s}_{j_3})Z_{t_4}(\boldsymbol{s}_{j_4})e^{i\boldsymbol{s}_{j_3}\mathbf{\Omega}_{\boldsymbol{k}_3}-i\boldsymbol{s}_{j_4}\mathbf{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3}}\Bigg] \\
&\quad=\widehat{A}+\widehat{B}+\widehat{C}
\end{aligned}\tag{50}$$

where,

$$\begin{aligned}
\widehat{A} &= \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\mathbf{\Omega}_{\boldsymbol{k}_1})\overline{g(\mathbf{\Omega}_{\boldsymbol{k}_3})}\frac{1}{\prod_{j=1}^4 n_{t_j}}\sum_{\substack{j_1\neq j_2\\ j_3\neq j_4}}\mathrm{cov}\left[\delta_{t_1,j_1}Z_{t_1}(\boldsymbol{s}_{j_1})e^{i\boldsymbol{s}_{j_1}\mathbf{\Omega}_{\boldsymbol{k}_1}},\delta_{t_3,j_1}Z_{t_3}(\boldsymbol{s}_{j_1})e^{i\boldsymbol{s}_{j_3}\mathbf{\Omega}_{\boldsymbol{k}_3}}\right] \\
&\qquad\times\mathrm{cov}\left[\delta_{t_2,j_2}Z_{t_2}(\boldsymbol{s}_{j_2})e^{-i\boldsymbol{s}_{j_2}\mathbf{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}},\delta_{t_4,j_4}Z_{t_4}(\boldsymbol{s}_{j_4})e^{-i\boldsymbol{s}_{j_4}\mathbf{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3}}\right]
\end{aligned}$$

$$\begin{aligned}
\widehat{B} &= \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\mathbf{\Omega}_{\boldsymbol{k}_1})\overline{g(\mathbf{\Omega}_{\boldsymbol{k}_3})}\frac{1}{\prod_{j=1}^4 n_j}\sum_{\substack{j_1\neq j_2\\ j_3\neq j_4}}\mathrm{cov}\left[\delta_{t_1,j_1}Z_{t_1}(\boldsymbol{s}_{j_1})e^{i\boldsymbol{s}_{j_1}\mathbf{\Omega}_{\boldsymbol{k}_1}},\delta_{t_4,j_4}Z_{t_4}(\boldsymbol{s}_{j_4})e^{-i\boldsymbol{s}_{j_4}\mathbf{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3}}\right] \\
&\qquad\times\mathrm{cov}\left[\delta_{t_2,j_2}Z_{t_2}(\boldsymbol{s}_{j_2})e^{-i\boldsymbol{s}_{j_2}\mathbf{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}},\delta_{t_3,j_1}Z_{t_3}(\boldsymbol{s}_{j_1})e^{i\boldsymbol{s}_{j_3}\mathbf{\Omega}_{\boldsymbol{k}_3}}\right]
\end{aligned}$$

$$\widehat{C} = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}\frac{1}{\prod_{j=1}^4 n_j}\sum_{\substack{j_1\neq j_2\\ j_3\neq j_4}}\text{cum}\Big[\delta_{t_1,j_1}Z_{t_1}(\boldsymbol{s}_{j_1})e^{i\boldsymbol{s}_{j_1}\boldsymbol{\Omega}_{\boldsymbol{k}_1}},\delta_{t_2,j_2}Z_{t_2}(\boldsymbol{s}_{j_2})e^{-i\boldsymbol{s}_{j_2}\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1}},$$
$$\delta_{t_3,j_1}Z_{t_3}(\boldsymbol{s}_{j_1})e^{-i\boldsymbol{s}_{j_3}\boldsymbol{\Omega}_{\boldsymbol{k}_3}},\delta_{t_4,j_4}Z_{t_4}(\boldsymbol{s}_{j_4})e^{i\boldsymbol{s}_{j_4}\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3}}\Big].$$

Simplifications for these terms can be obtained by using the methods developed in Subba Rao [2015a]. Using this we can show

$$\widehat{A} = \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^d}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f_{t_3-t_1}(\boldsymbol{\Omega})\overline{f_{t_4-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1})}d\boldsymbol{\Omega} + R_{1,t_3-t_1,t_4-t_2},$$

$$\widehat{B} = \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^d}\int_{\mathcal{D}_{\boldsymbol{r}_1}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}_1})}f_{t_4-t_1}(\boldsymbol{\Omega})\overline{f_{t_3-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}d\boldsymbol{\Omega} + R_{2,t_4-t_1,t_3-t_2}$$

and,

$$\widehat{C} = \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{2d}}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)}f_{t_2-t_1,t_3-t_1,t_4-t_1}(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\boldsymbol{\Omega}_2,-\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1})d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 + R_{3,t_2-t_1,t_3-t_1,t_4-t_1},$$

where,

$$\begin{aligned}
|R_{1,t_3-t_1,t_4-t_2}| &= O(\rho_{t_3-t_1}\rho_{t_4-t_2}\ell_{\lambda,a,n}),\\
|R_{2,t_4-t_1,t_3-t_2}| &= O(\rho_{t_4-t_1}\rho_{t_3-t_2}\ell_{\lambda,a,n}), \text{ and}\\
R_{3,t_2-t_1,t_3-t_1,t_4-t_1} &= O\left(\rho_{t_2-t_1}\rho_{t_3-t_1}\rho_{t_4-t_1}\left[\ell_{\lambda,a,n}+\frac{(a\lambda)^d}{n^2}\right]\right).
\end{aligned}$$

We further observe that use of the expansions given in (50) to obtain an expression for $\text{var}[\widehat{a}_g(\omega_k;\boldsymbol{r}_1,r_2)]$ can make the notations extremely cumbersome and difficult to follow. Proofs which only involve DFTs can substantially reduce cumbersome notations. However, a DFT based proof requires the frequency grid to be bounded, and as mentioned in the discussion at the start of this section, $\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)$ and $\widetilde{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2)$ are only asymptotically equivalent if the frequency grid is bounded. Therefore to simplify notations, for the remainder of this section we focus on the case that the frequency grid is bounded. However, we mention that exactly the same bounds apply to the case when the frequency grid is unbounded.

We observe that in order to obtain an expression for $\lambda^d\text{cov}[\widehat{a}_g(\omega_{k_1};\boldsymbol{r}_1,r_2),\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_3,r_4)]$ (in the case that the frequency grid is bounded) we require the expansion

$$\lambda^d\text{cov}\left[\sum_{\boldsymbol{k}_1=-\boldsymbol{a}}^{\boldsymbol{a}} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})J_{t_1}(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})},\sum_{\boldsymbol{k}_3=-\boldsymbol{a}}^{\boldsymbol{a}} g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})\overline{J_{t_4}(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3})}\right] = \widetilde{A}+\widetilde{B}+\widetilde{C},$$

where

$$\widetilde{A} = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}\mathrm{cov}\left[J_{t_1}(\boldsymbol{\Omega}_{\boldsymbol{k}_1}), J_{t_3}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})\right]\mathrm{cov}\left[\overline{J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})}, \overline{J_{t_4}(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3})}\right],$$

$$\widetilde{B} = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}\mathrm{cov}\left[J_{t_1}(\boldsymbol{\Omega}_{\boldsymbol{k}_1}), \overline{J_{t_4}(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3})}\right]\mathrm{cov}\left[\overline{J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})}, J_{t_3}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})\right],$$

$$\widetilde{C} = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}\mathrm{cum}\left[J_{t_1}(\boldsymbol{\Omega}_{\boldsymbol{k}_1}), \overline{J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})}, \overline{J_{t_3}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}, J_{t_4}(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3})\right].$$

Now we obtain simplified expressions for $\widetilde{A}, \widetilde{B}$ and $\widetilde{C}$.

$$\widetilde{A} = \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^d}\int_{\mathcal{D}} |g(\boldsymbol{\Omega})|^2 f_{t_3-t_1}(\boldsymbol{\Omega})\overline{f_{t_4-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1})}d\boldsymbol{\Omega} + R_{1,t_3-t_1,t_4-t_2}, \tag{51}$$

$$\widetilde{B} = \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^d}\int_{\mathcal{D}_{\boldsymbol{r}_1}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}_1})} f_{t_4-t_1}(\boldsymbol{\Omega})\overline{f_{t_3-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}d\boldsymbol{\Omega} + R_{2,t_4-t_1,t_3-t_2}, \tag{52}$$

$$\widetilde{C} = \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{2d}}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)} f_{t_2-t_1,t_3-t_1,t_4-t_1}(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \boldsymbol{\Omega}_2, -\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1})d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 + R_{3,t_2-t_1,t_3-t_1,t_4-t_1}. \tag{53}$$

Comparing the above with (50), when the frequency grid is unbounded, see that the expressions are identical. We use the above to prove Lemma 4.2.

PROOF of Lemma 4.2 By decomposing the covariance we have

$$\lambda^d \mathrm{cov}\left[\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2), \widehat{a}_g(\omega_{k_4};\boldsymbol{r}_3,r_4)\right] = I_{k_2,k_4} + II_{k_2,k_4} + III_{k_2,k_4},$$

where,

$$I_{k_2,k_4} = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1},\omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_3},\omega_{k_4})\right] \times\mathrm{cov}\left[\overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2})}, \overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3},\omega_{k_4+r_4})}\right],$$

$$II_{k_2,k_4} = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}\mathrm{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1},\omega_{k_2}), \overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3},\omega_{k_4+r_4})}\right] \times\mathrm{cov}\left[\overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2})}, J(\boldsymbol{\Omega}_{\boldsymbol{k}_3},\omega_{k_4})\right],$$

and

$$III_{k_2,k_4} = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}$$
$$\times\text{cum}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1},\omega_{k_2}), \overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2})}, \overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_3},\omega_{k_4})}, J(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3},\omega_{k_4+r_4})\right].$$

By using (51)-(52) we obtain expressions for the I_{k_2,k_4}, II_{k_2,k_4} and III_{k_2,k_4}. We first consider I_{k_2,k_4}. Using (51) we have,

$$\begin{aligned} I_{k_2,k_4} &= \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}\text{cov}\left[J(\boldsymbol{\Omega}_{\boldsymbol{k}_1},\omega_{k_2}), J(\boldsymbol{\Omega}_{\boldsymbol{k}_3},\omega_{k_4})\right] \\ &\qquad\times\text{cov}\left[\overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1},\omega_{k_2+r_2})}, \overline{J(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3},\omega_{k_4+r_4})}\right] \\ &= I_{k_1,k_2,M} + I_{k_1,k_2,R}, \end{aligned} \tag{54}$$

where,

$$\begin{aligned} I_{k_1,k_2,M} &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T^2}\sum_{t_1,t_2,t_3,t_4=1}^{T}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f_{t_3-t_1}(\boldsymbol{\Omega})\overline{f_{t_4-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})} \\ &\qquad\times e^{it_1\omega_{k_2}-it_2\omega_{k_2+r_2}-it_3\omega_{k_3}+it_4\omega_{k_4+r_4}}d\boldsymbol{\Omega}, \\ I_{k_1,k_2,R} &= \frac{1}{(2\pi)^{d+2}T^2}\sum_{t_1,t_2,t_3,t_4=1}^{T} R_{1,t_3-t_1,t_4-t_2}e^{it_1\omega_{k_2}-it_2\omega_{k_2+r_2}-it_3\omega_{k_3}+it_4\omega_{k_4+r_4}}. \end{aligned}$$

We first find an expression for $I_{k_1,k_2,M}$

$$\begin{aligned} I_{k_1,k_2,M} &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T^2}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2\left(\sum_{s_1=-(T-1)}^{T-1} f_{s_1}(\boldsymbol{\Omega})e^{-is_1\omega_{k_4}}\sum_{t_1=1}^{T-|s_1|}e^{it_1(\omega_{k_4}-\omega_{k_2})}\right)\times \\ &\qquad\left(\sum_{s_2=-(T-1)}^{T-1}\overline{f_{s_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}e^{is_2\omega_{k_4+r_2}}\sum_{t_2=1}^{T-|s_2|}e^{it_2(\omega_{k_2+r_2}-\omega_{k_4+r_4})}\right)d\boldsymbol{\Omega} \\ &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{k_2=k_4}I_{r_2=r_4}}{(2\pi)^d}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f(\boldsymbol{\Omega},\omega_{k_2})\overline{f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{k_2+r_2})}d\boldsymbol{\Omega} + O\left(\frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{T}+\ell_{\lambda,a,n}\right) \\ &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{k_2=k_4}I_{r_2=r_4}}{(2\pi)^d}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f(\boldsymbol{\Omega},\omega_{k_2})f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{k_2+r_2})d\boldsymbol{\Omega} + O\left(\frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{T}+\ell_{\lambda,a,n}\right). \end{aligned}$$

It is straightforward to show that $I_{k_1,k_2,R} = O(\ell_{\lambda,a,n})$. Therefore we have

$$I_{k_1,k_2} = \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{k_2=k_4}I_{r_2=r_4}}{(2\pi)^d}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f(\boldsymbol{\Omega},\omega_{k_2})f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{k_2+r_2})d\boldsymbol{\Omega} + O\left(\frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{T}+\ell_{\lambda,a,n}\right).$$

Using the same arguments and (52)

$$
\begin{aligned}
II_{k_2,k_4} &= \frac{1}{\lambda^d}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_2})}\sum_{t_1,\ldots,t_4=1}^{T}\exp(it_1\omega_{k_2}+it_4\omega_{k_4+r_4}-it_2\omega_{k_2+r_2}-it_3\omega_{k_4}) \\
&\quad \mathrm{cov}\left[J_{t_1}(\boldsymbol{\Omega}_{\boldsymbol{k}_1}),\overline{J_{t_4}(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3})}\right]\mathrm{cov}\left[\overline{J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})},J_{t_3}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})\right] \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T^2}\int_{\mathcal{D}_{\boldsymbol{r}_1}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}_1})}\left(\sum_{s_1=-(T-1)}^{T-1} f_{s_1}(\boldsymbol{\Omega})e^{is_1\omega_{k_2}}\sum_{t_1=1}^{T-|s_1|}e^{it_1(\omega_{k_4}+\omega_{k_2}+\omega_{r_2})}\right) \\
&\quad\times\left(\sum_{s_2=-(T-1)}^{T-1}\overline{f_{s_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1})}e^{-is_2\omega_{k_2+r_2}}\sum_{t_3=1}^{T-|s_2|}e^{it_3(\omega_{k_2}+\omega_{r_4}+\omega_{k_4})}\right)d\boldsymbol{\Omega}+O(\ell_{\lambda,a,n}) \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{k_4=T-k_2-r_2}I_{r_2=r_4}}{(2\pi)^d}\int_{\mathcal{D}_{\boldsymbol{r}_1}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}_1})}f(\boldsymbol{\Omega},-\omega_{k_2})f(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{k_2+r_2})d\boldsymbol{\Omega} \\
&\quad+O\left(\frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{T}+\ell_{\lambda,a,n}\right). \qquad (55)
\end{aligned}
$$

Using (53) (see the proof of Theorem 4.1, Jentsch and Subba Rao [2015] for details) we have

$$
\begin{aligned}
& III_{k_2,k_4} \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{2d}T^2}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)}\sum_{t_1,t_2,t_3,t_4=1}^{T} f_{t_2-t_1,t_3-t_1,t_4-t_1}(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\boldsymbol{\Omega}_2,-\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_2}) \\
&\quad\times e^{it_1\omega_{k_2}-it_2\omega_{k_2+r_2}-it_3\omega_{k_4}+it_4\omega_{k_4+r_4}}d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 \\
&\qquad +\frac{1}{(2\pi)^d T^2}\sum_{t_1,t_2,t_3,t_4=1}^{T} R_{3,t_2-t_1,t_3-t_1,t_4-t_1}e^{it_1\omega_{k_2}-it_2\omega_{k_2+r_2}-it_3\omega_{k_4}+it_4\omega_{k_4+r_4}} \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{2d}T^2}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)}\sum_{s_1,s_2,s_2=-(T-1)}^{T-1} f_{s_1,s_2,s_3}(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\boldsymbol{\Omega}_2,-\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1}) \\
&\quad\times e^{is_1\omega_{k_2+r_2}+is_2\omega_{k_4}-is_3\omega_{k_4+r_4}}\sum_{t=|\min(s_i,0)|+1}^{T-|\max(s_i,0)|}e^{it(\omega_{k_2}-\omega_{k_2+r_2}-\omega_{k_4}+\omega_{k_4+r_4})}d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 \\
&\quad+\frac{1}{(2\pi)^{2d}T^2}\sum_{s_1,s_2,s_2=-(T-1)}^{T-1} R_{3,s_1,s_2,s_3}e^{is_1\omega_{k_2+r_2}+is_2\omega_{k_4}-is_3\omega_{k_4+r_4}} \\
&\quad\times\sum_{t=|\min(s_i,0)|+1}^{T-|\max(s_i,0)|}e^{it(\omega_{k_2}-\omega_{k_2+r_2}-\omega_{k_4}+\omega_{k_4+r_4})}.
\end{aligned}
$$

By changing the limits of the sum we have

$$
\begin{aligned}
& III_{k_2,k_4} \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{2d}T^2}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)} \sum_{s_1,s_2,s_2=-(T-1)}^{T-1} f_{s_1,s_2,s_3}(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\boldsymbol{\Omega}_2,-\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1}) \\
&\qquad \times e^{-is_1\omega_{k_2+r_2}-is_2\omega_{k_4}+is_3\omega_{k_4+r_4}} \sum_{t=1}^{T} e^{it(\omega_{k_2}-\omega_{k_2+r_2}-\omega_{k_4}+\omega_{k_4+r_4})} d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 + O\left(\ell_{\lambda,a,n}+\frac{1}{T^2}\right) \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}}{(2\pi)^{2d}T}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)} f(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{k_2}+\omega_{r_2},\boldsymbol{\Omega}_1,\omega_{k_2},-\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1},-\omega_{k_4}-\omega_{r_2}) \\
&\qquad d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 + O\left(\frac{\ell_{\lambda,a,n}}{T}+\frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}}{T^2}\right)
\end{aligned} \tag{56}
$$

The above results imply

$$
\begin{aligned}
& \lambda^d \mathrm{cov}[\widehat{a}_g(\omega_{k_2};\boldsymbol{r}_1,r_2),\widehat{a}_g(\omega_{k_4};\boldsymbol{r}_3,r_4)] \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}}{(2\pi)^d}\Bigg(I_{k_2=k_4}\int_{\mathcal{D}} g(\boldsymbol{\Omega})\overline{g(\boldsymbol{\Omega})} f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{k_2+r_2}) f(\boldsymbol{\Omega},\omega_{k_2}) d\boldsymbol{\Omega} \\
&\qquad + I_{k_4=T-k_2-r_2}\int_{\mathcal{D}_{\boldsymbol{r}}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}_1})} f(\boldsymbol{\Omega},-\omega_{k_2}) f(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{k_2+r_2}) d\boldsymbol{\Omega}\Bigg) \\
&\qquad + O\left(\ell_{\lambda,a,n}+\frac{1}{T}\right).
\end{aligned}
$$

By using the well known identities

$$
\begin{aligned}
\mathrm{cov}(\Re A,\Re B) &= \frac{1}{2}\big(\Re\mathrm{cov}(A,B)+\Re\mathrm{cov}(A,\bar{B})\big) \\
\mathrm{cov}(\Im A,\Im B) &= \frac{1}{2}\big(\Re\mathrm{cov}(A,B)-\Re\mathrm{cov}(A,\bar{B})\big), \\
\mathrm{cov}(\Re A,\Im B) &= \frac{-1}{2}\big(\Im\mathrm{cov}(A,B)-\Im\mathrm{cov}(A,\bar{B})\big),
\end{aligned} \tag{57}
$$

we immediately obtain (20).

Asymptotic normality is proved using sufficient mixing assumptions. □

A.2.1 PROOF of results in Section 7.1 (used in Section 4.2)

We start by analyzing the sampling properties of the first test statistic $\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2)$.

PROOF of Lemma 7.1 We first note that

$$
\frac{\lambda^d T}{2}\mathrm{cov}\left[\widehat{A}_{g,h}(\boldsymbol{r}_1,r_2),\widehat{A}_{g,h}(\boldsymbol{r}_3,r_4)\right] = I+II+III,
$$

where,

$$I = \frac{2}{T}\sum_{k_2,k_4=1}^{T/2} I_{k_2,k_4} \quad II = \frac{2}{T}\sum_{k_2,k_4=1}^{T/2} II_{k_2,k_4} \quad III = \frac{2}{T}\sum_{k_2,k_4=1}^{T/2} III_{k_2,k_4}$$

and I_{k_2,k_4}, II_{k_2,k_4} and III_{k_2,k_4} are defined in the proof of Lemma 4.2. We now obtain expressions for these terms. By substituting the expression for I_{k_2,k_4} in (54) into I we have

$$\begin{aligned} I &= \frac{2I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T}\sum_{t_1,t_2,t_3,t_4=1}^{T}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f_{t_3-t_1}(\boldsymbol{\Omega})\overline{f_{t_4-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}d\boldsymbol{\Omega} \\ &\quad \times e^{-it_2\omega_{r_2}+it_4\omega_{r_4}}\left(\frac{1}{T^2}\sum_{k_2,k_4=1}^{T/2} h(\omega_{k_2})\overline{h(\omega_{k_4})}e^{i\omega_{k_2}(t_1-t_2)}e^{-i\omega_{k_4}(t_3-t_4)}\right) \\ &\quad + \frac{2}{T^3}\sum_{k_2,k_4=1}^{T/2} h(\omega_{k_2})\overline{h(\omega_{k_4})}\sum_{t_1,t_2,t_3,t_4=1} R_{1,t_3-t_1,t_4-t_2}e^{it_1\omega_{k_2}-it_2\omega_{k_2+r_2}-it_3\omega_{k_4}it_4\omega_{k_4+r_4}} \\ &= I_M + I_R. \end{aligned}$$

We first obtain a neat expression for the leading term I_M. Using that the function $h:[0,\pi]\to\mathbb{R}$ is piecewise Lipschitz continuous and the integral approximation of the Riemann sum, we have

$$\frac{2}{T}\sum_{k=1}^{T/2} h(\omega_k)e^{ij\omega_k} = h_j + O(T^{-1})$$

where $h_j = \frac{1}{\pi}\int_0^\pi h(\omega)e^{ij\omega}d\omega$ and the Fourier coefficients decay at the rate $|h_j| \leq C|j|^{-1}I(j\neq 0)$. This approximation gives

$$\begin{aligned} &\frac{4}{(2\pi)^2T^2}\sum_{k_2,k_4=1}^{T/2} h(\omega_{k_2})\overline{h(\omega_{k_4})}\exp(i\omega_{k_2}(t_1-t_2))\exp(-i\omega_{k_4}(t_3-t_4)) \\ &\qquad = h_{t_1-t_2}\overline{h_{t_3-t_4}} + O(h_{t_1-t_2}T^{-1} + h_{t_3-t_4}T^{-1} + T^{-2}). \end{aligned}$$

Substituting this into I_M and using that $|h_j| \leq C|j|^{-1}I(j\neq 0)$ gives

$$\begin{aligned} I_M &= \frac{2I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T}\sum_{t_1,t_2,t_3,t_4=1}^{T} h_{t_1-t_2}\overline{h_{t_3-t_4}}e^{-it_2\omega_{r_2}+it_4\omega_{r_4}}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f_{t_3-t_1}(\boldsymbol{\Omega})\overline{f_{t_4-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}d\boldsymbol{\Omega} \\ &\quad + O((\log T)T^{-1}). \end{aligned}$$

By making the following change of variables, $s_1 = t_3 - t_1$, $s_2 = t_4 - t_2$ and $s_3 = t_1 - t_2$ (so

$t_3 - t_4 = s_1 - s_2 + s_3$) we have

$$
\begin{aligned}
I_M &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T}\sum_{s_1,s_2,s_3,t_2} h_{s_3}\overline{h_{s_1-s_2+s_3}} e^{is_2\omega_{r_4}} e^{-it_2(\omega_{r_2}-\omega_{r_4})}\int_{\mathcal{D}} |g(\boldsymbol{\Omega})|^2 f_{s_1}(\boldsymbol{\Omega})\overline{f_{s_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1})}d\boldsymbol{\Omega} \\
&\quad +O((\log T)T^{-1}) \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4}}{(2\pi)^{d+2}}\sum_{s_1,s_2,s_3} h_{s_3}\overline{h_{s_1-s_2+s_3}} e^{is_2\omega_{r_2}}\int_{\mathcal{D}} |g(\boldsymbol{\Omega})|^2 f_{s_1}(\boldsymbol{\Omega})\overline{f_{s_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1})}d\boldsymbol{\Omega} \\
&\quad +O((\log T)T^{-1}),
\end{aligned}
$$

where in the last term we have used that $T^{-1}\sum_{t=1}^{T} e^{-it_2(\omega_{r_2}-\omega_{r_4})} = I(r_1 = r_2)$. Next we use that $\sum_{s_3} h_{s_3}\overline{h_{s_3+(s_1-s_2)}} = \frac{1}{\pi}\int |h(\omega)|^2 \exp(-i\omega(s_1-s_2))d\omega$ to give

$$
\begin{aligned}
I_M &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4}}{\pi(2\pi)^{d+2}}\int_0^{\pi} |h(\omega)|^2 \sum_{s_1,s_2} e^{-i\omega(s_1-s_2)} e^{is_2\omega_{r_2}}\int_{\mathcal{D}} |g(\boldsymbol{\Omega})|^2 f_{s_1}(\boldsymbol{\Omega})\overline{f_{s_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1})}d\boldsymbol{\Omega}d\omega \\
&\quad +O((\log T)T^{-1}) \\
&= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4}}{\pi(2\pi)^{d}}\int_0^{\pi}\int_{\mathcal{D}} |h(\omega)|^2 |g(\boldsymbol{\Omega})|^2 f(\boldsymbol{\Omega},\omega)\overline{f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega+\omega_{r_2})}d\boldsymbol{\Omega}, d\omega \\
&\quad +O((\log T)T^{-1})
\end{aligned}
$$

By using a similar method we can show that $|I_R| = O(\ell_{\lambda,a,n})$. Altogether (using that f is real) we get

$$
\begin{aligned}
I &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4}}{(2\pi)^{d}}\int_0^{\pi}\int_{\mathcal{D}} |h(\omega)|^2 |g(\boldsymbol{\Omega})|^2 f(\boldsymbol{\Omega},\omega) f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega+\omega_{r_2})d\boldsymbol{\Omega}d\omega \\
&\quad +O((\log T)T^{-1}+\ell_{\lambda,a,n}) \\
&= I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4}\int_0^{\pi} |h(\omega)|^2 V_g(\omega;\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2})d\omega.
\end{aligned}
$$

Using similar arguments we can show that

$$
\begin{aligned}
II &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T}\sum_{t_1,t_2,t_3,t_4=1}^{T}\int_{\mathcal{D}_{\boldsymbol{r}_1}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}})} f_{t_4-t_1}(\boldsymbol{\Omega})\overline{f_{t_3-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}d\boldsymbol{\Omega} \\
&\quad \times e^{it_2\omega_{r_2}+it_4\omega_{r_4}}\left(\frac{1}{T^2}\sum_{k_2,k_4=1}^{T/2} h(\omega_{k_2})\overline{h(\omega_{k_4})} e^{i\omega_{k_2}(t_1-t_2)} e^{-i\omega_{k_4}(t_3-t_4)}\right) + O(\ell_{\lambda,a,n}).
\end{aligned}
$$

We set $s_1 = t_4 - t_1$, $s_2 = t_3 - t_2$, $s_3 = t_1 - t_2$ (and $t_3 - t_4 = s_2 - s_1 - s_3$) to give

$$\begin{aligned} II &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T} \sum_{s_1,s_2,s_3,t_2} h_{s_3}\overline{h_{s_2-s_1-s_3}} \int_{\mathcal{D}_{\boldsymbol{r}_1}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}})} f_{s_1}(\boldsymbol{\Omega}) f_{s_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}}) d\boldsymbol{\Omega} \\ &\quad \times e^{it_2\omega_{r_2}+i(s_1+s_3+t_2)\omega_{r_4}} + O(\ell_{\lambda,a,n} + (\log T)T^{-1}). \end{aligned}$$

By changing the limits of the sum over t_2 we have

$$\begin{aligned} II &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T} \sum_{s_1,s_2,s_3} h_{s_3}\overline{h_{s_2-s_1-s_3}} \int_{\mathcal{D}_{\boldsymbol{r}_1}} g(\boldsymbol{\Omega})\overline{g(-\boldsymbol{\Omega}-\boldsymbol{\Omega}_{\boldsymbol{r}})} f_{s_1}(\boldsymbol{\Omega}) f_{s_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}}) d\boldsymbol{\Omega} \\ &\quad \times e^{i(s_1+s_3)\omega_{r_4}} \underbrace{\sum_{t_2=1}^{T} e^{it_2(\omega_{r_1}+\omega_{r_2})}}_{r_1=T-r_2} + O(\ell_{\lambda,a,n} + (\log T)T^{-1}) = O(\ell_{\lambda,a,n} + (\log T)T^{-1}), \end{aligned}$$

where the last line follows from the fact that r_1 and r_2 are constrained such that $0 \leq r_1 \leq r_2 < T/2$. The following expression for III follows immediately from (56).

$$\begin{aligned} III &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4}}{\pi^2(2\pi)^{2d}} \int_0^\pi \int_0^\pi \int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)} h(\omega_1)\overline{h(\omega_2)} \\ &\quad \times f_4(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_1+\omega_{r_2}, \boldsymbol{\Omega}_2, \omega_2, -\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1}, -\omega_2-\omega_{r_2}) d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 d\omega_1 d\omega_2 \\ &\qquad + O((\log T)T^{-1} + \ell_{\lambda,a,n}). \end{aligned}$$

This gives us

$$\begin{aligned} &\frac{\lambda^d T}{2} \mathrm{cov}\left[\widehat{A}_{g,h}(\boldsymbol{r}_1, r_2), \widehat{A}_{g,h}(\boldsymbol{r}_3, r_4)\right] \\ &= I_{\boldsymbol{r}_1=\boldsymbol{r}_3} I_{r_2=r_4} \Bigg(\frac{1}{\pi} \int_0^\pi |h(\omega)|^2 V_g(\omega; \boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_{r_2}) d\omega + \frac{1}{(2\pi)^{2d}\pi^2} \int_0^\pi \int_0^\pi \int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)} \\ &\quad \times h(\omega_1)\overline{h(\omega_2)} f_4(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1}, \omega_1+\omega_{r_2}, \boldsymbol{\Omega}_2, \omega_2, -\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1}, -\omega_2-\omega_{r_2}) d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 d\omega_1 d\omega_2 \Bigg) \\ &\quad + O((\log T)T^{-1} + \ell_{\lambda,a,n}). \end{aligned}$$

Note that $1/[(2\pi)^d\pi^2] = 4/(2\pi)^{2d+2}$ gives the fourth order cumulant term in (42).

By using the expressions for I, II and III and (57), we obtain (42).

By using mixing-type arguments the CLT can be proved. □

A.2.2 Proof of results in Section 7.1 (used in Section 4.3)

PROOF of Lemma 7.2 equation (43) Expanding $\mathrm{cov}\left[B_{g,h;H}(\omega_{j_1 H}; \boldsymbol{r}_1, r_2), B_{g,h;H}(\omega_{j_2 H}; \boldsymbol{r}_3, r_4)\right]$

gives

$$\lambda^d H \text{cov}\left[B_{g,h;H}(\omega_{j_1 H};\boldsymbol{r}_1,r_2), B_{g,h;H}(\omega_{j_2 H};\boldsymbol{r}_3,r_4)\right] = I_H + II_H + III_H,$$

where

$$\begin{aligned} I_H &= \frac{1}{H}\sum_{k_2,k_4=1}^{H} I_{j_1H+k_2,j_2H+k_4}, \\ II_H &= \frac{1}{H}\sum_{k_2,k_4=1}^{H} II_{j_1H+k_2,j_2H+k_4}, \\ III_H &= \frac{1}{H}\sum_{k_2,k_4=1}^{H} III_{j_1H+k_2,j_2H+k_4}, \end{aligned}$$

and I_{k_2,k_4}, II_{k_2,k_4} and III_{k_2,k_4} are defined in the proof of Lemma 4.2. We now find expressions for these terms, first focusing on the case $j_1 = j_2 = j$. By using (55) we have,

$$I_H = I_{H,M} + I_{H,R},$$

where

$$\begin{aligned} I_{H,M} &= \frac{H I_{\boldsymbol{r}_1=\boldsymbol{r}_3}}{(2\pi)^{d+2}T^2}\sum_{t_1,t_2,t_3,t_4=1}^{T}\int_{\mathcal{D}} |g(\boldsymbol{\Omega})|^2 f_{t_3-t_1}(\boldsymbol{\Omega})\overline{f_{t_4-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}d\boldsymbol{\Omega} \times e^{-it_2\omega_{r_2}+it_4\omega_{r_4}} \\ &\quad \times\left(\frac{1}{H^2}\sum_{k_2,k_4=1}^{H} h(\omega_{jH}+\omega_{k_2})\overline{h(\omega_{jH}+\omega_{k_4})}e^{i(\omega_{jH}+\omega_{k_2})(t_1-t_2)}e^{-i(\omega_{jH}+\omega_{k_4})(t_3-t_4)}\right) \\ I_{H,R} &= \frac{1}{T^2 H}\sum_{k_2,k_4=1}^{H} h(\omega_{jH+k_2})\overline{h(\omega_{jH+k_4})} \\ &\qquad \sum_{t_1,t_2,t_3,t_4=1} R_{1,t_3-t_1,t_4-t_2} e^{it_1\omega_{jH+k_2}-it_2\omega_{jH+k_2+r_2}-it_3\omega_{jH+k_4}+it_4\omega_{jH+k_4+r_4}}. \end{aligned}$$

We first bound the inner sum in $I_{H,M}$. Using the approximation of the Riemann sum by an integral we have,

$$\frac{1}{H}\sum_{k=1}^{H} h(\omega_{jH}+\omega_k)e^{is\omega_k} = \frac{T}{H}\int_{\omega_{jH}}^{\omega_{(j+1)H}} h(\omega)e^{is\omega}d\omega + O(H^{-1}) = h_{s,H}(\omega_{jH}) + O(H^{-1}). \tag{58}$$

Applying the above to the following product gives

$$\frac{1}{H^2}\sum_{k_2,k_4=1}^{H} h(\omega_{jH}+\omega_{k_2})\overline{h(\omega_{jH}+\omega_{k_4})}e^{i(\omega_{jH}+\omega_{k_2})(t_1-t_2)}e^{-i(\omega_{jH}+\omega_{k_4})(t_3-t_4)}$$
$$= h_{t_1-t_2,H}(\omega_{jH})\overline{h_{t_3-t_4,H}(\omega_{jH})}+O(h_{t_1-t_2,H}(\omega_{jH})H^{-1}+h_{t_3-t_4,H}(\omega_{jH})H^{-1}+H^{-2}).$$

Substituting the above into $I_{H,M}$, using that

$$\frac{H}{(2\pi)^{d+2}T^2}\sum_{t_1,t_2,t_3,t_4=1}^{T}\int_{\mathcal{D}}|g(\boldsymbol{\Omega})|^2 f_{t_3-t_1}(\boldsymbol{\Omega})\overline{f_{t_4-t_2}(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}})}d\boldsymbol{\Omega}=O(H)$$

and the same arguments used to bound I_M in the proof of Lemma 7.1 we have,

$$\begin{aligned} I_{H,M} &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}T}{(2\pi)^{d+2}H}\int_{2\pi\omega_{jH}}^{2\pi\omega_{(j+1)H}}\int_{\mathcal{D}}|h(\omega)|^2|g(\boldsymbol{\Omega})|^2 f(\boldsymbol{\Omega},\omega)f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}},\omega+\omega_{r_2})d\boldsymbol{\Omega}d\omega \\ &\quad +O(H^{-1}+(\log T)T^{-1}). \end{aligned}$$

Using the same argument we can show that $I_{H,R}=O(\ell_{\lambda,a,n})$, which gives altogether

$$\begin{aligned} I_H &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}T}{(2\pi)^{d+2}H}\int_{\omega_{jH}}^{\omega_{(j+1)H}}\int_{\mathcal{D}}|h(\omega)|^2|g(\boldsymbol{\Omega})|^2 f(\boldsymbol{\Omega},\omega)f(\boldsymbol{\Omega}+\boldsymbol{\Omega}_{\boldsymbol{r}},\omega+\omega_{r_2})d\boldsymbol{\Omega}d\omega \\ &\quad +O(H^{-1}+(\log T)T^{-1}+\ell_{\lambda,a,n}). \end{aligned}$$

Using the same methods, we can show that $II_H=O(H^{-1}+(\log T)T^{-1}+\ell_{\lambda,a,n})$ (since $\leq r_2,r_4\leq T/2$). Finally to bound III_H we substitute (56) into III_H to give

$$\begin{aligned} III_H &= \frac{I_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}}{(2\pi)^{2d}TH}\sum_{k_2,k_4=1}^{H}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)}\times \\ &\quad f(\boldsymbol{\Omega}_1,\omega_{jH+k_2},-\boldsymbol{\Omega}_1-\boldsymbol{\Omega}_{\boldsymbol{r}_1},-\omega_{jH+k_2+r_2},-\boldsymbol{\Omega}_2,-\omega_{jH+k_4})d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 \\ &\qquad +O\left(\frac{H\ell_{\lambda,a,n}}{T}+\frac{HI_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}}{T^2}\right) \end{aligned}$$

By using (58) we have

$$\begin{aligned} III_H &= \frac{TI_{\boldsymbol{r}_1=\boldsymbol{r}_3}I_{r_2=r_4}}{H(2\pi)^{2d+2}}\int_{\omega_{jH}}^{\omega_{(j+1)H}}\int_{\omega_{jH}}^{\omega_{(j+1)H}}\int_{\mathcal{D}^2} g(\boldsymbol{\Omega}_1)\overline{g(\boldsymbol{\Omega}_2)}h(\omega_1)\overline{h(\omega_2)} \\ &\quad \times f_4(\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_{\boldsymbol{r}_1},\boldsymbol{\Omega}_2,\omega_2,-\boldsymbol{\Omega}_2-\boldsymbol{\Omega}_{\boldsymbol{r}_1},-\omega_2-\omega_{r_2})d\boldsymbol{\Omega}_1 d\boldsymbol{\Omega}_2 d\omega_1 d\omega_2 \\ &\qquad +O((\log T)T^{-1}+\ell_{\lambda,a,n}+H^{-1}). \end{aligned}$$

We observe that $III_H=O(H/T)$. Thus by using (57) we obtain (43) and a similar expression for the imaginary parts. □

PROOF of Lemma 7.2 equation (44) The proof of (44) follows immediately from (42). □

Finally we consider the sampling properties of $\widehat{D}_{g,h,v;H}(\boldsymbol{r}_1, r_2)$.

PROOF of Lemma 7.3. To prove (45) we expand the expectation squared in terms of covariance and expectations to give

$$\mathrm{E}[\lambda^d D_{g,h,v;H}(\boldsymbol{r}_1, r_2)] = I + II$$

where

$$\begin{aligned} I &= \frac{2H}{2T}\sum_{j=0}^{(T/2H)-1}\mathrm{var}[\sqrt{H\lambda^d}B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)], \text{ and} \\ II &= \frac{2H}{2T}\sum_{j=0}^{(T/2H)-1}\left|\mathrm{E}[\sqrt{\lambda^d H}B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)]\right|^2. \end{aligned}$$

Using (43) we have

$$\begin{aligned} I &= \frac{2H}{T}\sum_{j=0}^{T/(2H)-1}\frac{2}{2v(\omega_{j_1H})}\bigg(\mathrm{var}[\sqrt{H\lambda^d}\Re B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)] \\ &\quad +\mathrm{var}[\sqrt{H\lambda^d}\Im B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)]\bigg) + O(\ell_{\lambda,a,n}) \\ &= \frac{2H}{T}\sum_{j=0}^{T/(2H)-1}\frac{1}{v(\omega_{j_1H})}W_{g,h}(\omega_{jH};\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2}) + O\left(\ell_{\lambda,a,n}+\frac{1}{H}\right) \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{W_{g,h}(\omega;\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2})}{v(\omega)}d\omega + O\left(\ell_{\lambda,a,n}+\frac{1}{H}+\frac{H}{T}\right) \\ &= E_{g,h,v}(\boldsymbol{\Omega}_{\boldsymbol{r}_1},\omega_{r_2}) + O\left(\ell_{\lambda,a,n}+\frac{1}{H}+\frac{H}{T}\right). \end{aligned}$$

Next we consider the second term II. First considering the expectation we note that

$$\mathrm{E}[\sqrt{\lambda^d H}B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)] = \frac{\sqrt{\lambda^{d/2}}}{\sqrt{H}}\sum_{k=1}^{H}h(\omega_{jH+k})\mathrm{E}[\widehat{a}_g(\omega_{jH+k};\boldsymbol{r}_1,r_2)].$$

By using Lemma 4.1 we obtain bounds on $\mathrm{E}[\widehat{a}_g(\omega_{jH+k};\boldsymbol{r}_1,r_2)]$, however, these rely on the number of zeros in $\boldsymbol{r}_1$ and whether r_2 is zero or not. More precisely,

$$\left|\frac{\sqrt{\lambda^{d/2}}}{\sqrt{H}}\sum_{k=1}^{H}h(\omega_{jH+k})\mathrm{E}[\widehat{a}_g(\omega_{jH+k};\boldsymbol{r}_1,r_2)]\right| = O\left(\frac{\lambda^{d/2}H^{1/2}\prod_{j=1}^{d-b}(\log\lambda+\log|m_j|)}{T^{I_{r_2-r_4\neq 0}}\lambda^{d-b}}\right).$$

Therefore,

$$II = O\left(\frac{\lambda^d H[\prod_{j=1}^{d-b}(\log\lambda + \log|m_j|)]^2}{(T^{I_{r_2-r_4\neq 0}}\lambda^{d-b})^2}\right) = o(1).$$

This proves (45).

To prove (46) we expand the covariance in terms of products cumulants to give

$$\begin{aligned}
&\frac{T}{2M}\mathrm{cov}\left[\lambda^d D_{g,h,v;H}(\boldsymbol{r}_1, r_2), \lambda^d D_{g,h,v;H}(\boldsymbol{r}_3, r_4)\right] \\
&= \frac{2\lambda^{2d}H}{T}\sum_{j_1,j_2=0}^{(T/2H)-1}\frac{1}{v(\omega_{j_1H})\overline{v(\omega_{j_2H})}}\Bigg(\left|\mathrm{cov}[B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2), B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2)]\right|^2 \\
&\quad + \left|\mathrm{cov}[B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2), \overline{B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2)}]\right|^2 \\
&\quad + \mathrm{cum}\left[B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2), \overline{B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)}, B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2), \overline{B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2)}\right] \\
&\quad + \mathrm{E}[B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)]\mathrm{cum}\left[\overline{B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2)}, B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2), \overline{B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2)}\right] \\
&\quad + \text{similar terms involving the product of third and first order cumulants}\Bigg).
\end{aligned}$$

By using that

$$\begin{aligned}
&\frac{1}{\lambda^{3d}}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3,\boldsymbol{k}_5=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})g(\boldsymbol{\Omega}_{\boldsymbol{k}_5})\mathrm{cum}\Big[J_{t_1}(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})}, J_{t_3}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})\overline{J_{t_4}(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3})}, \\
&J_{t_5}(\boldsymbol{\Omega}_{\boldsymbol{k}_5})\overline{J_{t_6}(\boldsymbol{\Omega}_{\boldsymbol{k}_5+\boldsymbol{r}_5})}\Big] = O\left(\sum_{\mathcal{B}_3}\prod_{(t_i,t_j)\in\mathcal{B}_3}\rho_{t_i-t_j}\frac{\log^{3d}(a)}{\lambda^{2d}}\right)
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
&\frac{1}{\lambda^{4d}}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_3,\boldsymbol{k}_5,\boldsymbol{k}_7=-a}^{a} g(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\Omega}_{\boldsymbol{k}_3})g(\boldsymbol{\Omega}_{\boldsymbol{k}_5})}g(\boldsymbol{\Omega}_{\boldsymbol{k}_7})\mathrm{cum}\Big[J_{t_1}(\boldsymbol{\Omega}_{\boldsymbol{k}_1})\overline{J_{t_2}(\boldsymbol{\Omega}_{\boldsymbol{k}_1+\boldsymbol{r}_1})}, \overline{J_{t_3}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}J_{t_4}(\boldsymbol{\Omega}_{\boldsymbol{k}_3+\boldsymbol{r}_3}), \\
&J_{t_5}(\boldsymbol{\Omega}_{\boldsymbol{k}_5})\overline{J_{t_6}(\boldsymbol{\Omega}_{\boldsymbol{k}_5+\boldsymbol{r}_1})}, \overline{J_{t_7}(\boldsymbol{\Omega}_{\boldsymbol{k}_3})}J_{t_8}(\boldsymbol{\Omega}_{\boldsymbol{k}_7+\boldsymbol{r}_3})\Big] = O\left(\sum_{\mathcal{B}_4}\prod_{(t_i,t_j)\in\mathcal{B}_4}\rho_{t_i-t_j}\frac{\log^{4d}(a)}{\lambda^{3d}}\right),
\end{aligned} \tag{60}$$

where $\mathcal{B}_3$ and $\mathcal{B}_4$ denotes the set of all pairwise indecomposable partitions of the sets $\{1,2,3\}\times\{4,5,6\}$ and $\{1,2,3,4\}\times\{5,4,6,7\}$ (for example, it contains the element $(1,4),(3,6)$

, (5, 8), (2, 7)) respectively, we can show that

$$\lambda^{3d/2}\mathrm{cum}\left[a_g(\omega_{k_2};\boldsymbol{r}_1,r_2),\overline{a_g(\omega_{k_2};\boldsymbol{r}_1,r_2)},\overline{a_g(\omega_{k_4};\boldsymbol{r}_1,r_2)}\right]=O\left(\frac{\log^{3d}(a)}{\lambda^{d/2}}\right)$$
$$\lambda^{2d}\mathrm{cum}\left[a_g(\omega_{k_2};\boldsymbol{r}_1,r_2),\overline{a_g(\omega_{k_2};\boldsymbol{r}_1,r_2)},\overline{a_g(\omega_{k_4};\boldsymbol{r}_1,r_2)},a_g(\omega_{k_4};\boldsymbol{r}_1,r_2)\right]=O\left(\frac{\log^{4d}(a)}{\lambda^{d}}\right).$$

From this we expect (by using the methods detailed in the proof of Lemma B.5, Eichler [2008]), though a formal proof is not given, that the terms involving cumulants of order three and above are asymptotically negligible. Moreover that $\left|\mathrm{cov}[B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2),\overline{B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2)}]\right|^2$ is asymptotically negligible for $j_1\neq j_2$. Using this we have

$$\begin{aligned}&\frac{T}{2M}\mathrm{cov}\left[\lambda^d D_{g,h,v;H}(\boldsymbol{r}_1,r_2),\lambda^d D_{g,h,v;H}(\boldsymbol{r}_3,r_4)\right]\\ =\;&\frac{2\lambda^{2d}H}{T}\sum_{j_1,j_2=0}^{(T/2H)-1}\frac{1}{v(\omega_{j_1H})\overline{v(\omega_{j_2H})}}\left|\mathrm{cov}[B_{g,h;H}(\omega_{j_1H};\boldsymbol{r}_1,r_2),B_{g,h;H}(\omega_{j_2H};\boldsymbol{r}_1,r_2)]\right|^2.\end{aligned}$$

Substituting (43) into the above gives (46). □