Submitted to the Annals of Statistics
arXiv: arXiv:0000.0000

SUPPLEMENT TO "A FREQUENCY DOMAIN EMPIRICAL LIKELIHOOD METHOD FOR IRREGULARLY SPACED SPATIAL DATA"

BY SOUTIR BANDYOPADHYAY*, SOUMENDRA N. LAHIRI† AND DANIEL J. NORDMAN‡

Lehigh University, North Carolina State University and Iowa State University

We present some details of the proofs and some additional simulation results for the main paper.

1. Proofs of the lemmas. For completeness, we restate the lemmas and give the proofs. The proof of Proposition 4.1, from Section 4 of the manuscript, is deferred to the end here. The notation and notational conventions correspond to those of the main paper. To avoid confusion with the equation numbers in the main, the equation numbers in this section are given as (S.∗).

We require some additional notation. Let $C_n(\boldsymbol{\omega})$ and $S_n(\boldsymbol{\omega})$ denote the cosine and the sine transforms of the data, respectively given by the real and the imaginary parts of $d_n(\boldsymbol{\omega})$ (cf. (2.2)). Define the bias corrected periodogram $\tilde{I}_n(\boldsymbol{\omega}) = I_n(\boldsymbol{\omega}) - n^{-1}\lambda_n^d \hat{\sigma}_n(\mathbf{0})$ and its variant $I_n^*(\boldsymbol{\omega}) = I_n(\boldsymbol{\omega}) - n^{-1}\lambda_n^d \sigma(\mathbf{0})$. Let $\hat{f}(\boldsymbol{\omega}) = \int e^{\iota \mathbf{x}' \boldsymbol{\omega}} f(\mathbf{x}) d\mathbf{x}$, $\boldsymbol{\omega} \in \mathbb{R}^d$ and similarly define $\widehat{f^2}$. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. For $r \in \mathbb{N}$, let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ denote the standard basis of $\mathbb{R}^r$, with $\mathbf{e}_i \in \mathbb{R}^r$ having a 1 in the ith position and 0 elsewhere. Next, for $r \in \mathbb{N}$, define the joint cumulant of random variables $Y_1, \cdots, Y_r$ by

$$\chi_r(Y_1, \cdots, Y_r) = \frac{\partial^r}{\partial t_1, \ldots \partial t_r} \log E \exp\Big(\iota[t_1 Y_1 + \ldots + t_r Y_r]\Big)\Big|_{t_1 = \ldots = t_r = 0}.$$

We extend this definition to complex valued random variables $Z_i = Y_{1i} + \iota Y_{2i}$, $i = 1, \ldots, r$ by multilinearity, by setting, $\chi_r(Z_1, \cdots, Z_r) = \chi_r(Z_1, \cdots, Z_{i-1}, Y_{1i}, Z_{i+1}, \cdots, Z_r) + \iota \chi_r(Z_1, \cdots, Z_{i-1}, Y_{2i}, Z_{i+1}, \cdots, Z_r)$ for all i.

*Research partially supported by NSF DMS-1406622.
†Research partially supported by NSF DMS-1007703 and DMS-1310068.
‡Research partially supported by NSF DMS-1406747.
AMS 2000 subject classifications: Primary 62M30; secondary 62E20
Keywords and phrases: Confidence sets, Discrete Fourier Transform, Estimating equations, Hypotheses testing, Periodogram, Spectral moment conditions, Stochastic design, Variogram, Wilks' Theorem

Recall that, for a random quantity T depending on both $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ and $\{\mathbf{X}_1, \mathbf{X}_2, \ldots\}$, ET denotes the conditional expectation of T given $\mathbb{X} \equiv \{\mathbf{X}_1, \mathbf{X}_2, \ldots\}$ and P similarly denotes conditional probability. Similarly, in the following, $\chi_r\Big(I_n(\boldsymbol{\omega}_1), \ldots, I_n(\boldsymbol{\omega}_r)\Big)$ refers to the conditional cumulant of $I_n(\boldsymbol{\omega}_1), \ldots, I_n(\boldsymbol{\omega}_r)$, given $\mathbb{X}$. Again write $P_{\mathbf{X}}$ and $E_{\mathbf{X}}$ to denote the probability and the expectation under the joint distribution of $\mathbf{X}_1, \mathbf{X}_2, \ldots$. We let C or $C(\cdot)$ denote generic constants that depend on their arguments (if any), but do not depend on n or the $\{\mathbf{X}_i\}$. Similarly, let $P_k(\cdot)$ denote a generic polynomial of degree $k \geq 1$ with real co-efficients that do not depend on n and the $\{\mathbf{X}_i\}$.

Lemma 7.1 first provides an integral bound on sinusoids summed over the frequency grid $\mathcal{N}$ from (3.2) (cf. Section 3.1). This technical result is used in the proof of Lemma 7.5 and Proposition 4.1 to follow.

LEMMA 7.1. *For the frequency grid* $\Big\{\mathbf{j}\lambda_n^{-\kappa} : \mathbf{j} \in \mathbb{Z}^d, \mathbf{j} \in [-C^*\lambda_n^\eta, C^*\lambda_n^\eta]^d\Big\} \equiv \{\boldsymbol{\omega}_{kn}\}_{k=1}^N$ *from (3.2) and any* $\epsilon > 0$,

$$\int \Big|\sum_{k=1}^N \exp(\iota \mathbf{t}'\boldsymbol{\omega}_{kn})\Big|(1 + \|\mathbf{t}\|)^{-d(1+\epsilon)} d\mathbf{t} \leq C(d,\epsilon)[\lambda_n^\kappa \log \lambda_n]^d.$$

Proof For a real number x, write $(x)_{2\pi}$ for x modulo 2π with values in $[-\pi, \pi)$, i.e., $(x)_{2\pi} = x - 2\pi k$ for all $x \in [2\pi k - \pi, 2\pi k + \pi)$, $k \in \mathbb{Z}$. Then, as the frequencies lie on a regular rectangular grid,

$$\begin{aligned} &\sum_{k=1}^N \exp(\iota \mathbf{t}'\boldsymbol{\omega}_{kn}) \\ =\ & \prod_{j=1}^d \Big\{ \sum_{-C^*\lambda_n^\eta \leq k \leq C^*\lambda_n^\eta} \exp(\iota t_j k \lambda_n^{-\kappa}) \Big\} \\ =\ & \prod_{j=1}^d \Big[\frac{1 - \exp(\iota t_j \lambda_n^{-\kappa}\lfloor C^*\lambda_n^\eta\rfloor)}{1 - \exp(\iota t_j\lambda_n^{-\kappa})} + \frac{1 - \exp(-\iota t_j \lambda_n^{-\kappa}\lfloor C^*\lambda_n^\eta\rfloor)}{1 - \exp(-\iota t_j\lambda_n^{-\kappa})} - 1\Big]. \end{aligned}$$

Now using the bounds that $|\exp(\iota x) - [1 + \iota x]| \leq x^2/2$ for $|x| \leq 1$ and that $\inf\{|1 - \exp(\iota x)| : 1 \leq |x| \leq \pi\} > 0$, one can show that for any $t \in \mathbb{R}$,

$$\begin{aligned} &\Big|\frac{1 - \exp(\iota t \lambda_n^{-\kappa}\lfloor C^*\lambda_n^\eta\rfloor)}{1 - \exp(\iota t\lambda_n^{-\kappa})}\Big| \\ \leq\ & g_n(t) \equiv \begin{cases} (2C^* + 1)\lambda_n^\eta & \text{if} \quad |(t\lambda_n^{-\kappa})_{2\pi}| \leq \lambda_n^{-\eta} \\ \frac{C^*}{|(t\lambda_n^{-\kappa})_{2\pi}|} & \text{if} \quad \lambda_n^{-\eta} \leq |(t\lambda_n^{-\kappa})_{2\pi}| \leq \pi. \end{cases} \end{aligned} \tag{S.1}$$

Hence, it follows that

$$
\begin{aligned}
&\int \Big|\sum_{k=1}^{N} \exp(\iota \mathbf{t}'\boldsymbol{\omega}_{kn})\Big| (1+\|\mathbf{t}\|)^{-d(1+\epsilon)} d\mathbf{t} \\
\le\ & C(d) \prod_{j=1}^{d} \int_{-\infty}^{\infty} \Big\{ \lambda_n^{\eta} \mathbb{1}\Big(|(t_j \lambda_n^{-\kappa})_{2\pi}| \le \lambda_n^{-\eta} \Big) \\
& \qquad + |(t_j\lambda_n^{-\kappa})_{2\pi}|^{-1} \mathbb{1}\Big(\lambda_n^{-\eta} \le |(t_j\lambda_n^{-\kappa})_{2\pi}| \le \pi \Big) \Big\} (1+|t_j|)^{-1-\epsilon} dt_j \\
\le\ & C(d) \Big[\sum_{j\in\mathbb{Z}} \Big\{ \lambda_n^{\eta} \int_{|t\lambda_n^{-\kappa} - 2\pi j| \le \lambda_n^{-\eta}} (1+|t|)^{-1-\epsilon} dt \\
& \qquad + \int_{\lambda_n^{-\eta} \le |t\lambda_n^{-\kappa} - 2\pi j| \le \pi} |t\lambda_n^{-\kappa} - 2\pi j|^{-1} (1+|t|)^{-1-\epsilon} dt \Big\} \Big]^d \\
\le\ & C(d) \Big[\sum_{j\in\mathbb{Z}} \Big\{ \lambda_n^{\eta} \lambda_n^{-\eta+\kappa} (1 + 2\pi |j| \lambda_n^{\kappa})^{-(1+\epsilon)} \\
& \qquad + \int_{\lambda_n^{-\eta+\kappa} < |t| \le \pi\lambda_n^{\kappa}} \lambda_n^{\kappa} |t|^{-1} \Big(1 + |t + 2\pi j \lambda_n^{\kappa}| \Big)^{-(1+\epsilon)} dt \Big]^d \\
\le\ & C(d,\epsilon) \Big[\lambda_n^{\kappa} + \lambda_n^{\kappa} \sum_{j\in\mathbb{Z}} \Big\{ \Big(1 + \pi |j| \lambda_n^{\kappa} \Big)^{-(1+\epsilon)} \int_{\lambda_n^{-\eta+\kappa} < |t| \le \pi\lambda_n^{\kappa}} |t|^{-1} dt \Big\} \Big]^d \\
\le\ & C(d,\epsilon) \Big[\lambda_n^{\kappa} \log \lambda_n \Big]^d .
\end{aligned}
$$

This completes the proof of Lemma 7.1. □

Lemma 7.2 next establishes bounds on cumulants involving general spatial averages. This result is applied to develop for expansions of the bias and variance of the periodogram $I_n(\cdot)$ in Lemma 7.3 to follow and is also used in the proof of Lemma 7.5.

LEMMA 7.2. *Let $\{W(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ be a possibly nonstationary strongly mixing random field, with finite means (i.e., $EW(\mathbf{s}) \in \mathbb{R}$) and mixing coefficient $\alpha(\cdot,\cdot)$, that is independent of $\mathbb{X}$. Also, let Condition (C.2)(i) hold. Then for any $r \in \mathbb{N}, r \ge 2$ and $\delta \in (0,1]$,*

$$
\begin{aligned}
&\sup_{\mathbf{b}_1,\cdots,\mathbf{b}_r \in [-1,1]^n} \left| \chi_r \left(\left[\sum_{j=1}^{n} b_{1j} W(\mathbf{s}_j) \right], \cdots, \left[\sum_{j=1}^{n} b_{rj} W(\mathbf{s}_j) \right] \right) \right| \\
&\le C(r,\delta) \left[\sum_{k=0}^{C\lambda_n} k^{(r-1)d} \left[\alpha(k;r) \right]^{\delta/(r+\delta)} \right] \zeta_{r+\delta}\ n^r \lambda_n^{-(r-1)d} \quad a.s.\ (P_{\mathbf{X}}),
\end{aligned}
$$

where $\mathbf{b}_k = (b_{k1}, \ldots, b_{kn})' \in [-1,1]^n$, $1 \le k \le r$, *and where* $\zeta_a = \sup\{(E|W(\mathbf{s}) - EW(\mathbf{s})|^a)^{\frac{1}{a}} : \mathbf{s} \in \mathbb{R}^d\}$, $a \ge 1$.

Proof This is a consequence of Lemma 5.1 of [BLN], though we outline the main steps for completeness. Let $m_n = n\lambda_n^{-d}$, $n \ge 1$ and recall the sampling region $\mathcal{D}_n = \lambda_n \mathcal{D}_0 \subset \mathbb{R}^d$. Let $J_n = \{\boldsymbol{j} \in \mathbb{Z}^d : \{\boldsymbol{j} + (0,1]^d\} \cap \mathcal{D}_n \neq \emptyset\}$. Based on the exponential inequality of [2] (Lemma 5.1) for the independent sampling site locations $\{\mathbf{s}_i = \lambda_n \mathbf{X}_i\}_{i=1}^n$, there exists a $C \in (0,\infty)$ such that

$$P_{\mathbf{X}}\left(\max_{\boldsymbol{j} \in J_n} \sum_{i=1}^n \mathbb{I}(\lambda_n \mathbf{X}_i \in \{\boldsymbol{j} + (0,1]^d\} \cap \mathcal{D}_n) > Cm_n \text{ infinitely often}\right) = 0,$$

where $\mathbb{I}(\cdot)$ denotes the indicator function. Hence, eventually for large n (a.s.$(P_{\mathbf{X}})$), the number of observations in $\{\boldsymbol{j} + (0,1]^d\} \cap \mathcal{D}_n$ is at most Cm_n for any $\boldsymbol{j} \in J_n$. For each $1 \le k \le r$, we then group the sums $\sum_{i=1}^n b_{ki} W(\mathbf{s}_i)$ corresponding to $\mathbf{s}_i$'s in each cube $\boldsymbol{j} + (0,1]^d$, $\boldsymbol{j} \in J_n$, as $\sum_{i=1}^n b_{ki} W(\mathbf{s}_i) = \sum_{\boldsymbol{j} \in J_n} \tilde{W}_k(\boldsymbol{j})$ for $\tilde{W}_k(\boldsymbol{j}) \equiv \sum_{i=1}^n b_{ki} W(\mathbf{s}_i) \mathbb{I}(\mathbf{s}_i \in \boldsymbol{j} + (0,1]^d)$. The observations $\tilde{W}(\boldsymbol{j})$, $\boldsymbol{j} \in J_n$, are now lattice variables (each a sum of no more than Cm_n $W(\cdot)$ variables). Applying Theorem 1.4.1.1 of [1], we can bound $|\chi_r\left(\sum_{\boldsymbol{j} \in J_n} \tilde{W}_k(\boldsymbol{j}), \cdots, \sum_{\boldsymbol{j} \in J_n} \tilde{W}_r(\boldsymbol{j})\right)|$ by

$$|J_n| C(r,\delta) \left[\sum_{k=0}^{C\lambda_n} k^{(r-1)d} \left[\alpha(k;r)\right]^{\delta/(r+\delta)}\right] \cdot \zeta_{r+\delta} \cdot (m_n)^r$$

using the mixing coefficient and covariance bounds based on $\alpha(\cdot;r)$ and $r+\delta$ moments of $\{W(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$; the value r in $\alpha(\cdot;r)$ owes to the fact that cumulants involve variables $\tilde{W}_k(\boldsymbol{j}_k)$, $\boldsymbol{j}_k \in J_n$, $1 \le k \le r$ defined on regions as unions of cubes with volume not exceeding r; see Section 4.1 for details on the mixing coefficient. Note the sum with $\alpha(\cdot;r)$ is over all possible (integer-valued) distances in ℓ_1-norm (i.e., $\|\cdot\|_1$) between cubes indexed by J_n, with a maximal distance of $C\lambda_n$ for some factor of λ_n. Because $|J_n| = O(|\mathcal{D}_n|)$ is of order of the volume $O(\lambda_n^d)$ of the sampling region $\mathcal{D}_n$, the result then follows. □

For notational simplicity, we shall suppose that the indexing of the elements of the set $\mathcal{N}$ is done in such a way that $\boldsymbol{\omega}_{1n} = \mathbf{0}$. Also, recall that we use the order symbols $o^u(\cdot)$ and $O^u(\cdot)$ if the corresponding bound is valid uniformly. To state the next lemma, define

$$H_n(\mathbf{x},\mathbf{t}) = \sigma(\mathbf{t}) f(\mathbf{x}) \left[f(\mathbf{x} + \lambda_n^{-1}\mathbf{t}) - f(\mathbf{x})\right], \quad \mathbf{x},\mathbf{t} \in \mathbb{R}^d$$

and set

$$R_n^{[1]}(\boldsymbol{\omega}) = 2 \int \int e^{-\iota t' \boldsymbol{\omega}} H_n(\mathbf{x}, \mathbf{t}) d\mathbf{t} d\mathbf{x}.$$

Then, we have the following result on the bias and the variance of the periodogram $I_n(\cdot)$ at the non-zero frequencies. The expansions in Lemma 7.3 are used to establish distributional properties of the periodogram, and certain sums of the periodogram, in the remaining lemmas here (Lemmas 7.4-7.7).

LEMMA 7.3. *Suppose that Conditions (C.0)-(C.1) and (C.2)(i) hold and that* $0 < \kappa < 1$. *Then, for any* $\epsilon \in (0, 1)$,

$$\text{(S.2)} \qquad \Big| EI_n(\boldsymbol{\omega}_{jn}) - \Big[A_n(\boldsymbol{\omega}_{jn}) + R_n^{[1]}(\boldsymbol{\omega}_{jn})\Big]\Big| = O^u(n^{-1/2+\epsilon}), \quad \textit{and}$$

$$\Big| Var(I_n(\boldsymbol{\omega}_{jn})) - \Big[A_n^2(\boldsymbol{\omega}_{jn}) + A_n(\boldsymbol{\omega}_{jn})P_1(D_n(\boldsymbol{\omega}_{jn})) + P_2(D_n(\boldsymbol{\omega}_{jn}))\Big]\Big|$$
$$\text{(S.3)} \qquad = \Big[A_n(\boldsymbol{\omega}_{jn}) + \lambda_n^{-1}\Big] \cdot O^u(n^{-1/2+\epsilon}) + O^u(\lambda_n^{-d} + n^{-1+2\epsilon}),$$

a.s. $(P_{\mathbf{X}})$, *for all* $2 \le j \le N$ *where* $P_k(\cdot)$ *is a polynomial of degree* k *(with real coefficients that do not depend on* n*),* $k = 1, 2$, *and* $D_n(\boldsymbol{\omega}) = (R_n^{[1]}(\boldsymbol{\omega}), Ed_n^2(\boldsymbol{\omega}), Ed_n^2(-\boldsymbol{\omega}))$.

Proof Fix $\epsilon \in (0, 1)$. Then, using Lemma 5.2 of [BLN], one can show that uniformly over $\boldsymbol{\omega}, \boldsymbol{\omega}^* \in \{\boldsymbol{\omega}_{jn} : j = 1, 2, \ldots, N\}$,

$$\begin{aligned} & Ed_n(\boldsymbol{\omega})d_n(-\boldsymbol{\omega}^*) \\ = \;& \lambda_n^d n^{-2}\Big[n\sigma(\mathbf{0})\hat{f}(\lambda_n(\boldsymbol{\omega} - \boldsymbol{\omega}^*)) + \binom{n}{2}\lambda_n^{-d}(2\pi)^d\Big\{\phi(\boldsymbol{\omega}^*)\widehat{f^2}(\lambda_n[\boldsymbol{\omega} - \boldsymbol{\omega}^*]) \\ & +\phi(\boldsymbol{\omega})\widehat{f^2}(\lambda_n[\boldsymbol{\omega}^* - \boldsymbol{\omega}])\Big\}\Big] + R_n(\boldsymbol{\omega}, \boldsymbol{\omega}^*) + O(n^{-1/2+\epsilon}), \end{aligned} \qquad \text{(S.4)}$$

a.s.$(P_{\mathbf{X}})$, where $R_n(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = \int\int \Gamma_n(\mathbf{x}; \boldsymbol{\omega}, \boldsymbol{\omega}^*)H_n(\mathbf{x}, \mathbf{t})d\mathbf{t}d\mathbf{x}$, with $\Gamma_n(\mathbf{x}; \boldsymbol{\omega}, \boldsymbol{\omega}^*) = \Gamma_{1n}(\mathbf{x}; \boldsymbol{\omega}, \boldsymbol{\omega}^*) + \Gamma_{1n}(\mathbf{x}; \boldsymbol{\omega}^*, \boldsymbol{\omega})$, $\Gamma_{1n}(\boldsymbol{\omega}^*, \boldsymbol{\omega}) = e^{-\iota \mathbf{t}' \boldsymbol{\omega}^*} e^{\iota \lambda_n \mathbf{x}'(\boldsymbol{\omega} - \boldsymbol{\omega}^*)}$. Relation (S.2) readily follows from (S.4).

Next consider (S.3). Note that

$$\text{(S.5)} \qquad |H_n(\mathbf{x}, \mathbf{t})| \le C\lambda_n^{-1}\|\mathbf{t}\||\sigma(\mathbf{t})|f(\mathbf{x}) \le C\lambda_n^{-1}$$

for all $\mathbf{x}, \mathbf{t}$.

Using the identities '$C_n(\boldsymbol{\omega}) = [d_n(\boldsymbol{\omega})+d_n(-\boldsymbol{\omega})]/2$' and '$S_n(\boldsymbol{\omega}) = [d_n(\boldsymbol{\omega})-d_n(-\boldsymbol{\omega})]/(2\iota)$' and using (S.4), one can show that a.s. $P_{\mathbf{X}}$,

(S.6)
$$\left.\begin{aligned} EC_n^2(\boldsymbol{\omega}_{jn}) &= 2^{-1}\left[A_n(\boldsymbol{\omega}_{jn}) + R_n^{[1]}(\boldsymbol{\omega}_{jn})\right] + P_{11}(D_n(\boldsymbol{\omega}_{jn})) + O(n^{-1/2+\epsilon}), \\ ES_n^2(\boldsymbol{\omega}_{jn}) &= 2^{-1}\left[A_n(\boldsymbol{\omega}_{jn}) + R_n^{[1]}(\boldsymbol{\omega}_{jn})\right] + P_{12}(D_n(\boldsymbol{\omega}_{jn})) + O(n^{-1/2+\epsilon}) \\ EC_n(\boldsymbol{\omega}_{jn})S_n(\boldsymbol{\omega}_{jn}) &= \iota P_{13}(D_n(\boldsymbol{\omega}_{jn})) \end{aligned}\right\}$$

for some polynomials $P_{1j}(\cdot)$ of degree one (with real coefficients). Next note that for any zero mean random variables $Y_1, \ldots, Y_4$,

$$\begin{aligned} &\mathrm{Cov}(Y_1Y_2,\, Y_3Y_4) \\ &= \chi_4(Y_1, \ldots, Y_4) + \chi_2(Y_1, Y_3)\chi_2(Y_2, Y_4) + \chi_2(Y_1, Y_4)\chi_2(Y_2, Y_3). \end{aligned} \tag{S.7}$$

Now using the expression for $\mathrm{Var}(I_n(\boldsymbol{\omega}_j))$ as a weighted sum of $\mathrm{Var}(C_n^2(\boldsymbol{\omega}_j))$, $\mathrm{Var}(S_n^2(\boldsymbol{\omega}_j))$ and $\mathrm{Cov}(C_n^2(\boldsymbol{\omega}_j), S_n^2(\boldsymbol{\omega}_j))$ and using (S.7), (S.5), (S.4) and Lemma 7.2, after some lengthy and tedious algebra, one can establish (S.3). We omit the routine details. □

The final proofs of the main distributional results about the SFDEL method (e.g., chi-square limits in Theorems 5.1-5.3) depend on the use of Lemmas 7.4-7.7 to follow; proofs of Theorems 5.1-5.3 appear Section 7.2 of the manuscript.

LEMMA 7.4. *Under the Conditions (C.0)-(C.3) and (C.5)'*

$$\begin{aligned} &E\left[\sum_{k=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{kn})G_{\theta_0}(\boldsymbol{\omega}_{kn})' I_n^2(\boldsymbol{\omega}_{kn})\right] \\ =\ & 2\sum_{k=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{kn})G_{\theta_0}(\boldsymbol{\omega}_{kn})' A_n^2(\boldsymbol{\omega}_{kn}) + o(b_n^2) \quad a.s.(P_{\mathbf{X}}). \end{aligned}$$

Proof Fix $\epsilon \in (0,1)$. Without loss of generality, we may assume $p = 1$. Also, for notational simplicity, we write $\boldsymbol{\omega}_{kn} \equiv \boldsymbol{\omega}_k$ and drop the qualifier 'a.s.($P_{\mathbf{X}}$)' from the statements below. Using (S.4), one gets

$$\left|Ed_n^2(\boldsymbol{\omega}_k) - \left[A^{\ddagger}(\boldsymbol{\omega}_k) + R_n^{[2]}(\boldsymbol{\omega}_k)\right]\right| = O_p^u(n^{-1/2+\epsilon}), \tag{S.8}$$

for $2 \le k \le N$, where $A^{\ddagger}(\boldsymbol{\omega}) = c_n^{-1}\sigma(\mathbf{0})\hat{f}(2\lambda_n\boldsymbol{\omega}) + (2\pi)^d\phi(\boldsymbol{\omega})2^{-1}[\widehat{f^2}(2\lambda_n\boldsymbol{\omega}) +$

$\widehat{f^2}(-2\lambda_n\boldsymbol{\omega})]$ and $R_n^{[2]}(\boldsymbol{\omega}) = \int\int[2\cos(\mathbf{t}+2\lambda_n\mathbf{x})'\boldsymbol{\omega}]H_n(\mathbf{x},\mathbf{t})d\mathbf{x}d\mathbf{t}$. By Lemma 7.3,

$$\begin{aligned}
&E\left[\sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k) I_n^2(\boldsymbol{\omega}_k)\right] \\
&\quad = \sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)\Big[\mathrm{Var}\Big(I_n(\boldsymbol{\omega}_k)\Big) + \Big(EI_n(\boldsymbol{\omega}_k)\Big)^2\Big] \\
&\quad = \sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)\Big[\Big\{A_n^2(\boldsymbol{\omega}_k) + A_n(\boldsymbol{\omega}_k)P_1(D_n(\boldsymbol{\omega}_k)) + P_2(D_n(\boldsymbol{\omega}_k))\Big\} \\
&\qquad\qquad + [A_n(\boldsymbol{\omega}_k) + \lambda_n^{-1}]\cdot O^u(n^{-1/2+\epsilon}) + O^u(\lambda_n^{-d} + n^{-1+2\epsilon}) \\
&\qquad\qquad\qquad + \Big(A_n(\boldsymbol{\omega}_k) + R_n^{[1]}(\boldsymbol{\omega}_k) + O^u(n^{-1/2+\epsilon})\Big)^2\Big]
\end{aligned}$$

$$= 2\sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k) A_n^2(\boldsymbol{\omega}_k) + R_{11n} + R_{12n} + R_{13n}, \quad \text{(say)} \tag{S.9}$$

where R_{1kn}'s are remainder terms satisfying:

$$\begin{aligned}
|R_{11n}| &\leq \sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)\Big\{[A_n(\boldsymbol{\omega}_k) + \lambda_n^{-1}]\cdot O^u(n^{-1/2+\epsilon}) + O^u(\lambda_n^{-d} + n^{-1+2\epsilon})\Big\} \\
R_{12n} &= \sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k) A_n(\boldsymbol{\omega}_k) P_1(D_n(\boldsymbol{\omega}_k)), \\
R_{13n} &= \sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k) P_2(D_n(\boldsymbol{\omega}_k)),
\end{aligned}$$

for some generic polynomials $P_k(\cdot)$ of degree $k \in \{1, 2\}$, with real coefficients that do not depend on n. Note that by Condition (C.3)(i), (ii),

$$\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)\phi(\boldsymbol{\omega}_k) = O(\lambda_n^{\kappa d}). \tag{S.10}$$

Hence,

$$|R_{11n}| = O\Big(\Big\{\Big[Nc_n^{-1} + \lambda_n^{\kappa d}\Big] + N\lambda_n^{-1}\Big\} n^{-1/2+\epsilon}\Big) = o(b_n^2). \tag{S.11}$$

By (S.5) and (S.10), $|R_{12n}| = O(\lambda_n^{\kappa d-1}) = o(b_n^2)$. Hence, it remains to show that $R_{13n} = o(b_n^2)$. By (S.8), the fact that $I_n(\mathbf{0}) = \lambda_n^d \bar{Z}_n^2$, and the arguments leading to (S.11), it is enough to show that

$$\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k) P_2\Big(R_n^{[1]}(\boldsymbol{\omega}_k), R_n^{[2]}(\boldsymbol{\omega}_k), A_n^{\ddagger}(\boldsymbol{\omega}_k)\Big) = o(b_n^2). \tag{S.12}$$

Note that by (C.5)',

$$\begin{aligned}
&\Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)[R_n^{[1]}(\boldsymbol{\omega}_k)]^2\Big| \\
= \;& \Big|\int\int\int\int \sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k) 4\exp\Big(\iota(\mathbf{t}+\mathbf{s})'\boldsymbol{\omega}_k\Big) \\
& \qquad\qquad H_n(\mathbf{t},\mathbf{x})H_n(\mathbf{s},\mathbf{y})d\mathbf{t}d\mathbf{x}d\mathbf{s}d\mathbf{y}\Big| \\
\le \;& C\lambda_n^{-2}\int\int \Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)\exp\Big(\iota(\mathbf{t}+\mathbf{s})'\boldsymbol{\omega}_k\Big)\Big| \|\mathbf{t}\|\|\mathbf{s}\||\sigma(\mathbf{t})|\sigma(\mathbf{s})|d\mathbf{t}d\mathbf{s} \\
\le \;& C\lambda_n^{-2}\zeta_{4+\delta}^4 \int\int M_n(\mathbf{t}+\mathbf{s})[\gamma_1(\mathbf{t})\gamma_1(\mathbf{s})]^{\frac{\delta}{4+\delta}} d\mathbf{t}d\mathbf{s} \\
= \;& o(b_n^2). \qquad \text{(S.13)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)[R_n^{[2]}(\boldsymbol{\omega}_k)]^2\Big| \\
\le \;& C\lambda_n^{-2}\int\int \Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)\exp\Big(\iota[(\mathbf{t}+\mathbf{s})+2\lambda_n(\mathbf{x}+\mathbf{y})]'\boldsymbol{\omega}_k\Big)\Big| \times \\
& \qquad\qquad \|\mathbf{t}\|\|\mathbf{s}\||\sigma(\mathbf{t})||\sigma(\mathbf{s})|f(\mathbf{x})f(\mathbf{y})d\mathbf{x}d\mathbf{y}d\mathbf{t}d\mathbf{s} \\
\le \;& C\zeta_{4+\delta}^4\lambda_n^{-2}\int\int M_n(\mathbf{t}+\mathbf{s}+2\lambda_n[\mathbf{x}+\mathbf{y}])\nu(d\mathbf{t},d\mathbf{x})\nu(d\mathbf{s},d\mathbf{y}) \\
= \;& o(b_n^2). \qquad \text{(S.14)}
\end{aligned}$$

Also, using (C.2)(ii) and (S.10), it is easy to verify that

$$\Big|\sum_{k=2}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_k)[A_n^{\ddagger}(\boldsymbol{\omega}_k)]^2\Big| = O(\lambda_n^{\kappa d})\cdot o^u(1) = o(b_n^2). \tag{S.15}$$

Now, using (S.13)-(S.15) and similar arguments for the cross-product terms, one gets (S.12). This completes the proof of Lemma 7.4. □

LEMMA 7.5. *Under the Conditions (C.0)-(C.3) and (C.5)',*

$$\sum_{i=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{in})G_{\theta_0}(\boldsymbol{\omega}_{in})'\Big[\tilde{I}_n^2(\boldsymbol{\omega}_{in}) - \Big(A_n^2(\boldsymbol{\omega}_{in}) + K^2\phi^2(\boldsymbol{\omega}_{in})\Big)\Big] = o_p(b_n^2), \quad a.s.(P_{\mathbf{X}}).$$

Proof Note that by Conditions (C.0)-(C.1) and (C.2)(i) above and by Lemma 5.2 of Lahiri [2], $\hat{\sigma}_n(\mathbf{0}) - \sigma(\mathbf{0}) = O_p(\lambda_n^{-d/2})$, a.s. $(P_{\mathbf{X}})$. Hence, it is enough to prove the lemma with $\tilde{I}_n(\cdot)$ replaced by $I_n^*(\cdot) \equiv I_n(\cdot) - c_n^{-1}\sigma(\mathbf{0})$. Note that

$$\sum_{i=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{in})G_{\theta_0}(\boldsymbol{\omega}_{in})'\Big(I_n(\boldsymbol{\omega}_{in}) - c_n^{-1}\sigma(\mathbf{0})\Big)^2 = \sum_{i=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{in})G_{\theta_0}(\boldsymbol{\omega}_{in})' I_n^2(\boldsymbol{\omega}_{in})$$

$$-2c_n^{-1}\sigma(\mathbf{0})\sum_{i=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{in})G_{\theta_0}'(\boldsymbol{\omega}_{in})I_n(\boldsymbol{\omega}_{in}) + c_n^{-2}[\sigma(\mathbf{0})]^2\sum_{i=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{in})G_{\theta_0}(\boldsymbol{\omega}_{in})'$$

$$\text{(S.16)} = J_{11} + J_{12} + J_{13} \quad \text{(say)}.$$

Clearly, J_{13} is deterministic but J_{11} and J_{12} are random. We now derive the 'in-probability, a.s.$(P_{\mathbf{X}})$-limits' of these two terms, starting with J_{11}.

For notational simplicity, for the rest of the proof, we again set $p = 1$. For $M \in [1, \infty)$, define $Z(\mathbf{s}; M) = Z(\mathbf{s})\mathbb{1}(|Z(\mathbf{s})| \leq M) - EZ(\mathbf{s})\mathbb{1}(|Z(\mathbf{s})| \leq M)$ and $Z^*(\mathbf{s}; M) = Z(\mathbf{s}) - Z(\mathbf{s}; M)$, $\mathbf{s} \in \mathbb{R}^d$. Also, let $d_n(\boldsymbol{\omega}; M)$ and $d_n^*(\boldsymbol{\omega}; M)$ be obtained from $d_n(\boldsymbol{\omega})$ by replacing $\{Z(\mathbf{s}_i) : i = 1, \ldots, n\}$ with $\{Z(\mathbf{s}_i; M) : i = 1, \ldots, n\}$ and $\{Z^*(\mathbf{s}_i; M) : i = 1, \ldots, n\}$, respectively. Note that $d_n^*(\boldsymbol{\omega}; M) = d_n(\boldsymbol{\omega}) - d_n(\boldsymbol{\omega}; M)$. Also, let $S_N(M) = b_n^{-2}\sum_{i=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})I_n^2(\boldsymbol{\omega}_{in}; M)$ and $S_N^*(M) = b_n^{-2}\sum_{i=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})I_n^{*2}(\boldsymbol{\omega}_{in}; M)$, where $I_n(\boldsymbol{\omega}; M) = |d_n(\boldsymbol{\omega}; M)|^2$ and $I_n^*(\boldsymbol{\omega}; M) = |d_n^*(\boldsymbol{\omega}; M)|^2$. Then, using Cauchy-Schwarz inequality and the inequality

$$\Big||z_1|^2 - |z_2|^2\Big| \leq (|z_1| + |z_2|)|z_1 - z_2| \text{ for any complex numbers } z_1, z_2,$$

one can show that

$$\Big|b_n^{-2}\sum_{i=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})\Big[I_n^2(\boldsymbol{\omega}_{in}) - I_n^2(\boldsymbol{\omega}; M)\Big]\Big| \leq \Big|S_N^*(M)\Big| + 2\Big|S_N(M)S_N^*(M)\Big|^{1/2}.$$

Hence, it suffices to show that, for some suitable sequence $M = M_n \to \infty$, a.s. $(P_{\mathbf{X}})$,

$$\text{(S.17)} \qquad \left.\begin{array}{rrcl} (i) & \lim_{n\to\infty} E\Big|S_N^*(M)\Big| & = & 0, \\ (ii) & \lim_{n\to\infty} |ES_N(M) - b_n^{-2}\Sigma_n| & = & 0, \\ (iii) & \lim_{n\to\infty} \operatorname{Var}\Big(S_N(M)\Big) & = & 0. \end{array}\right\}$$

We shall set $M_n = (\log \lambda_n)^{2d}$ for the rest of the proof.

First consider parts (i) and (ii) of (S.17). By arguments similar to those used in the proof of Lemma 7.4, we get

$$E(S_N(M)) = 2b_n^{-2} \sum_{i=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in}) A^2(\boldsymbol{\omega}_{in}; M) + o(1); \tag{S.18}$$

$$E\Big|S_N^*(M)\Big| = ES_N^*(M) = 2b_n^{-2} \sum_{i=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in}) A^{*2}(\boldsymbol{\omega}_{in}; M) + o(1), \tag{S.19}$$

a.s. $(P_{\mathbf{X}})$, provided the following analogs of (S.10) hold:

$$\Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{kn}) \phi^*(\boldsymbol{\omega}_{kn}; M)\Big| = o(\lambda_n^{\kappa d}); \tag{S.20}$$

$$\Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{kn}) \phi(\boldsymbol{\omega}_{kn}; M)\Big| = O(\lambda_n^{\kappa d}), \tag{S.21}$$

a.s. $(P_{\mathbf{X}})$. Here $A(\boldsymbol{\omega}; M)$ and $A^*(\boldsymbol{\omega}; M)$ are defined by replacing $\sigma(\cdot)$ and $\phi(\cdot)$ in $A(\boldsymbol{\omega})$ by the auto-covariance functions and the spectral densities of the $\{Z(\cdot; M)\}$- and $\{Z^*(\cdot; M)\}$-processes, respectively. (Note that for $p > 1$ and $1 \leq i \neq j \leq p$, by Cauchy-Schwarz inequality, $E|b_n^{-2} \sum_{k=1}^{N} G_{i,\theta_0}(\boldsymbol{\omega}_{kn}) G_{j,\theta_0}(\boldsymbol{\omega}_{kn}) I_n^{*2}(\boldsymbol{\omega}_{kn}, M)|$ admits a bound involving terms of the form $ES_N^*(M)$, and hence, the restricting attention to the $p = 1$ suffices.)

Let $\zeta_a^*(M) = \sup\{(E|Z^*(\mathbf{s}; M)|^a)^{1/a} : \mathbf{s} \in \mathbb{R}^d\}$ for $a > 0$. Also, write $\sigma^*(\mathbf{t}; M)$, $\sigma_1^*(\mathbf{t}; M)$ and $\sigma_2^*(\mathbf{t}; M)$ respectively for $\mathrm{Cov}(Z^*(\mathbf{0}; M), Z^*(\mathbf{t}; M))$, $\mathrm{Cov}(Z^*(\mathbf{0}; M), Z(\mathbf{t}; M))$ and $\mathrm{Cov}(Z(\mathbf{0}; M), Z^*(\mathbf{t}; M))$. Note that the bound by the monotone function h in Condition (C.3)(ii) need not hold for the spectral density $\phi^*(\cdot; M)$ (and for $\phi(\cdot; M)$) after truncation. We now develop an alternative approach to derive the growth bounds above. With $a = \delta/2$ (where $\delta \in (0, 1]$ is as in (C.1)), by Lemma 7.1 and the inversion formula, we have

$$
\begin{aligned}
& \Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{kn})\phi^*(\boldsymbol{\omega}_{kn};M)\Big| \\
\le\ & C(d)\sum_{k=1}^{N}\phi^*(\boldsymbol{\omega}_{kn};M) \\
\le\ & C(d)(2\pi)^{-d}\int\Big|\sum_{k=1}^{N}\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\Big||\sigma^*(\mathbf{t};M)|d\mathbf{t} \\
\le\ & C(d)\int\Big|\sum_{k=1}^{N}\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\Big|[\gamma_1(\|\mathbf{t}\|)]^{\frac{a}{2+a}}[\zeta_{2+a}^*(M)]^2 d\mathbf{t} \\
\le\ & C(d)\lambda_n^{\kappa d}(\log\lambda_n)^d\zeta_{4+\delta}^4 M^{-2} = o(\lambda_n^{\kappa d}), \qquad \text{(S.22)}
\end{aligned}
$$

where the step before the last one follows by Markov's inequality. This proves (S.20).

Next, consider (S.21). It is easy to verify that

$$
\sigma(\mathbf{t}) \;=\; \sigma(\mathbf{t};M)+\sigma^*(\mathbf{t};M)+\sigma_1^*(\mathbf{t};M)+\sigma_2^*(\mathbf{t};M).
$$

Now using arguments similar to those leading to (S.22), one can show that for $r=1,2$,

$$
\begin{aligned}
& \Big|\sum_{k=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{kn})\Big[[\phi(\boldsymbol{\omega}_{kn};M)]^r-[\phi(\boldsymbol{\omega}_{kn})]^r\Big]\Big| \\
\le\ & C(d,r)\int\Big|\sum_{k=1}^{N}\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\Big|\{|\sigma^*(\mathbf{t};M)|+\sigma_1^*(\mathbf{t};M)+\sigma_2^*(\mathbf{t};M)\}d\mathbf{t} \\
\le\ & C(d,r)\int\Big|\sum_{k=1}^{N}\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\Big|[\gamma_1(\|\mathbf{t}\|)]^{\frac{a}{2+a}}[\zeta_{2+a}^*(M)]^2\{[\zeta_{2+a}^*(M)]^2+\zeta_{2+a}^2\}d\mathbf{t} \\
=\ & o(\lambda_n^{\kappa d}). \qquad \text{(S.23)}
\end{aligned}
$$

This proves (S.21). Parts (i) and (ii) of (S.17) now follow from (S.18), (S.19), (S.22) and (S.23).

Next consider Part (iii) of (S.17). Write $C_n(\cdot;M)$ and $S_n(\cdot;M)$ for the real and the imaginary parts of $d_n(\cdot; M)$. Then,

$$\begin{aligned}
&\mathrm{Var}\Big(S_N(M)\Big) \\
&\quad b_n^{-4}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})G_{\theta_0}^2(\boldsymbol{\omega}_{jn})\chi_2\Big(I_n^2(\boldsymbol{\omega}_{in}; M), I_n^2(\boldsymbol{\omega}_{jn}; M)\Big), \\
&\quad = b_n^{-4}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})G_{\theta_0}^2(\boldsymbol{\omega}_{jn})\chi_2\Big(\big\{C_n^2(\boldsymbol{\omega}_{in}; M) + S_n^2(\boldsymbol{\omega}_{in}; M)\big\}^2, \\
&\qquad\qquad \big\{C_n^2(\boldsymbol{\omega}_{jn}; M) + S_n^2(\boldsymbol{\omega}_{jn}; M)\big\}^2\Big) \\
&\quad = Q_{1n}(M) + Q_{2n}(M) + \cdots + Q_{9n}(M). \text{ (say)}
\end{aligned}$$

We will show that $\lim_{M\to\infty}\limsup_{n\to\infty} Q_{1n}(M) = 0$, a.s. $(P_{\mathbf{X}})$, where

$$Q_{1n}(M) = b_n^{-4}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})G_{\theta_0}^2(\boldsymbol{\omega}_{jn})\chi_2\Big(C_n^4(\boldsymbol{\omega}_{in}; M), C_n^4(\boldsymbol{\omega}_{jn}; M)\Big).$$

The other terms can be shown to be negligible using similar arguments. Using the product formula of cumulants,

$$\chi_2\Big(C_n^4(\boldsymbol{\omega}_{in} : M), C_n^4(\boldsymbol{\omega}_{jn}; M)\Big) = \sum_{q=1}^{8}\sum\nolimits_q^{\star\star}\prod_{i=1}^{q}\chi_{|I_i|}\Big(C_n(I_i; M)\Big)$$

where $\sum_q^{\star\star}$ extends over all "indecomposable" partitions by q non-empty subsets $I_1, \cdots, I_q$ of the (2×4) array:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4). \end{matrix}$$

Since $EC_n(\boldsymbol{\omega}_{jn}; M) = 0$ for all j, the summands under $\sum_q^{\star\star}$ are all zero for $q = 5, \cdots, 8$. Hence it follows that

$$\begin{aligned}
&|Q_{1n}(M)| \\
&\le b_n^{-2}\sum\nolimits_4^{\star\star}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})G_{\theta_0}^2(\boldsymbol{\omega}_{jn})\left|\prod_{i=1}^{4}\chi_2\Big(C_n(I_i; M)\mathbb{1}(|I_i| = 2)\Big)\right| \\
&+ b_n^{-2}\sum_{q=1}^{3}\sum\nolimits_q^{\star\star}\sum_{i=1}^{N}\sum_{j=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in})G_{\theta_0}^2(\boldsymbol{\omega}_{jn}) \\
&\qquad \left|\prod_{1q}\chi_2\Big(C_n(I_i; M)\Big)\prod_{2q}\chi_{|I_i|}\Big(C_n(I_i; M)\Big)\right| \\
= \quad & Q_{1n}^{(1)}(M) + Q_{1n}^{(2)}(M), \text{ say,}
\end{aligned}$$

where the products $\prod_{1q}$ extends over all factors with $|I_i| = 2$ and $\prod_{2q}$ extends over the remaining factors (with $|I_i| \geq 3$). First consider $Q_{1n}^{(1)}(M)$. Now,

$$\begin{aligned} & Q_{1n}^{(1)}(M) \\ & \leq C b_n^{-4} \sum_{i=1}^{N} \sum_{j=1}^{N} G_{\theta_0}^2(\boldsymbol{\omega}_{in}) G_{\theta_0}^2(\boldsymbol{\omega}_{jn}) \Big[\chi_2^4\Big(C_n(\boldsymbol{\omega}_{in}; M), C_n(\boldsymbol{\omega}_{jn}; M)\Big) \\ & + \Big| \chi_2\Big(C_n(\boldsymbol{\omega}_{in}; M)^{(2)}\Big) \chi_2\Big(C_n(\boldsymbol{\omega}_{jn}; M)^{(2)}\Big) \Big| \chi_2^2\Big(C_n(\boldsymbol{\omega}_{in}; M), C_n(\boldsymbol{\omega}_{jn}; M)\Big) \Big], \end{aligned}$$

where, for any random variable W and $r \in \mathbb{N}$, we set $W^{(r)} = (W, \ldots, W)' \in \mathbb{R}^r$. Now using (S.6) and (S.8), and arguments in the proof of (S.21), one gets $\lim_{n\to\infty} Q_{1n}^{(1)}(M) = 0$, a.s. $(P_{\mathbf{X}})$.

Next, using Lemma 7.2 and similar arguments, one can show that each of the terms in $Q_{1n}^{(2)}(M)$ for $q = 1, 2, 3$ is negligible. Here we outline the main steps for the case $q = 1$ to point out the special treatment needed to handle cumulants of order greater than 4 under the moment and mixing conditions (C.0)-(C.1). Let $\zeta_a(M) = \sup\{(|Z(\mathbf{s}; M)|^a)^{1/a} : \mathbf{s} \in \mathbb{R}^d\}$, $a \geq 1$. Note that $\sum_{k=1}^{\infty} k^{3d} \gamma_1(k)^{\frac{\delta}{4+\delta}} < \infty$ implies that

$$\gamma_1(k) = o\Big([k^{-3d}]^{\frac{4+\delta}{\delta}}\Big) \quad \text{as} \quad k \to \infty.$$

Hence, for $q = 1$, by Lemma 7.2, for $\eta > 0$,

$$\begin{aligned} & \Big| \chi_8\Big(C_n(\boldsymbol{\omega}_{in}; M)^{(4)}, C_n(\boldsymbol{\omega}_{jn}; M)^{(4)}\Big) \Big| \\ \leq \; & C \left[\sum_{k=1}^{C\lambda_n} k^{7d} [\alpha(k; 8)]^{\frac{\eta}{8+\eta}} \right] \lambda_n^d [\lambda_n^{-d/2}]^8 \zeta_{8+\eta}^8(M) \\ \leq \; & C M^8 \lambda_n^{-3d} \sum_{k=1}^{C\lambda_n} k^{7d} \left[k^{-\frac{3d(4+\delta)}{\delta}} \right]^{\frac{\eta}{8+\eta}} \\ \leq \; & C(d, \eta) M^8 \lambda_n^{-3d}, \end{aligned}$$

by choosing $\eta > 0$ sufficiently large, such that $7 - \frac{15\eta}{8+\eta} < -1$, where the constant $C(d, \eta)$ does not depend on $i, j \in \{1, \ldots, N\}$. Using the bound above for $q = 1$ and similar arguments for the terms for $q = 2, 3$, one can conclude that $\lim_{n\to\infty} Q_{1n}^{(2)}(M) = 0$, a.s. $(P_{\mathbf{X}})$. This shows that (cf. (S.18))

$$\left\| b_n^{-2} \sum_{k=1}^{N} G_{\theta_0}(\boldsymbol{\omega}_{kn}) G_{\theta_0}(\boldsymbol{\omega}_{kn})' \Big[I_n^2(\boldsymbol{\omega}_{kn}) - 2A_n^2(\boldsymbol{\omega}_{kn}) \Big] \right\| = o_p(1), \quad \text{a.s.}(P_{\mathbf{X}}).$$

By similar arguments, $\|b_n^{-2}\sum_{k=1}^N G_{\theta_0}(\boldsymbol{\omega}_{kn})G_{\theta_0}(\boldsymbol{\omega}_{kn})'[I_n(\boldsymbol{\omega}_{kn})-A_n(\boldsymbol{\omega}_{kn})]\| = o_p(1)$, a.s.$(P_{\mathbf{X}})$. Hence, in view of (S.16), Lemma 7.5 follows. □

LEMMA 7.6. *Under the Conditions (C.0)-(C.3) and (C.5)', for any $\epsilon > 0$,*

$$P\left(\max_{1\le k\le N}\|G_{\theta_0}(\boldsymbol{\omega}_{kn})I_n(\boldsymbol{\omega}_{kn})\| > \epsilon b_n\right) = o(1), \ \textit{a.s.}\ (P_{\mathbf{X}}).$$

Proof W.l.g., suppose that $p = 1$. It is enough to show that a.s. $(P_{\mathbf{X}})$,

$$\begin{aligned}\lim_{n\to\infty} b_n^{-4}\sum_{i=1}^N G_{\theta_0}^4(\boldsymbol{\omega}_{in})\Big[EC_n^{*4}(\boldsymbol{\omega}_{in};M) + ES_n^{*4}(\boldsymbol{\omega}_{in};M)\Big] &= 0\\ \lim_{n\to\infty} b_n^{-8}\sum_{i=1}^N G_{\theta_0}^8(\boldsymbol{\omega}_{in})\Big[EC_n^{8}(\boldsymbol{\omega}_{in};M) + ES_n^{8}(\boldsymbol{\omega}_{in};M)\Big] &= 0,\end{aligned}$$

where $C_n^*(\cdot;M), C_n(\cdot;M),\ldots$ etc. are as defined in the proof of Lemma 7.5. Both of these relations can be proved by recasting the arguments in the proof of Lemma 7.5. We omit the routine details. □

LEMMA 7.7. *Let $ch^o(B)$ denote the interior of the convex hull of a set $B \in \mathbb{R}^p$. Under the Conditions (C.0)-(C.3) and (C.5)',*

$$P\Big(0 \in ch^o\{G_{\theta_0}(\boldsymbol{\omega}_{kn})\tilde{I}_n(\boldsymbol{\omega}_{kn})\}_{k=1}^N\Big) \to 1 \ \textit{as}\ n\to\infty \quad \textit{a.s.}\ (P_{\mathbf{X}}).$$

Proof We outline the main steps. Let $\mathcal{U} \equiv \{\boldsymbol{y} \in \mathbb{R}^p : \|\boldsymbol{y}\| = 1\}$. As in the proof of Lemma 7.5, it can be shown that, as $n\to\infty$ for any sequence $\boldsymbol{y}_n \in \mathcal{U}$ with $\boldsymbol{y}_n \to \boldsymbol{y}_0$ for some $\boldsymbol{y}_0 \in \mathcal{U}$,

$$\begin{aligned} d_n(\boldsymbol{y}_n) &\equiv \lambda_n^{-d\kappa}\sum_{j=1}^N \boldsymbol{y}_n' G_{\theta_0}(\boldsymbol{\omega}_{jn})\tilde{I}_n(\boldsymbol{\omega}_{jn})\mathbb{I}\left(\boldsymbol{y}_n' G_{\theta_0}(\boldsymbol{\omega}_{jn})\tilde{I}_n(\boldsymbol{\omega}_{jn}) > 0\right)\\ &\xrightarrow{p} \int_{\{\boldsymbol{y}_0' G_{\theta_0}\phi>0\}} \boldsymbol{y}_0' G_{\theta_0}(\boldsymbol{\omega})\phi(\boldsymbol{\omega})K\, d\boldsymbol{\omega} \quad \text{a.s.}(P_{\mathbf{X}}), \end{aligned} \tag{S.24}$$

where $\mathbb{I}(\cdot)$ denotes the indicator function. To ease the notation, we will suppress the qualification "a.s.$(P_{\mathbf{X}})$", which holds implicitly throughout the remainder. As $\int_{\mathbb{R}^d} G_{\theta_0}(\boldsymbol{\omega})\phi(\boldsymbol{\omega})\,d\boldsymbol{\omega} = 0$ by (C.2) and $\int_{\mathbb{R}^d} G_{\theta_0}(\boldsymbol{\omega})G_{\theta_0}'(\boldsymbol{\omega})\phi^2(\boldsymbol{\omega})\,d\boldsymbol{\omega}$ is positive definite by (C.3)(iv), $\Lambda \equiv \inf_{\boldsymbol{y}\in\mathcal{U}}\int_{\{\boldsymbol{y}' G_{\theta_0}\phi>0\}} G_{\theta_0}'(\boldsymbol{\omega})\boldsymbol{y}\phi(\boldsymbol{\omega})\,d\boldsymbol{\omega} \ge c_0$ holds for some $c_0 > 0$ (cf. Owen [3], Lemma 2). By (S.24) and using countability arguments, for any subsequence $\{n_j\}\subset\{n\}$, we take a further subsequence $\{n_k\} \subset \{n_j\}$, letting $n_k \equiv k$, such that $\liminf_{k\to\infty} \Lambda_k > c_0/2$ holds a.s.(P), for $\Lambda_k \equiv \inf_{\boldsymbol{y}\in\mathcal{U}} d_k(\boldsymbol{y})$. Hence, on this set of P-probability 1, it follows (pointwise) that $\Lambda_k > c_0/2 > 0$ eventually for large k. When $\Lambda_k > 0$ holds, 0 must lie in the interior convex hull of $\{G_{\theta_0}(\boldsymbol{\omega}_{jk})\tilde{I}_k(\boldsymbol{\omega}_{jk})\}_{j=1}^{N_k}$. If not, then by the separating hyperplane theorem, there exists $\boldsymbol{a} \in \mathcal{U}$ such that $\boldsymbol{a}'G_{\theta_0}(\boldsymbol{\omega}_{jk})\tilde{I}_k(\boldsymbol{\omega}_{jk}) \leq 0$ for all $j = 1, ..., N_k$, implying a contradiction $\Lambda_k \leq d_k(\boldsymbol{a}) = 0$. Thus, pointwise on a set of P-probability 1, 0 eventually belongs to $ch^0\{G_{\theta_0}(\boldsymbol{\omega}_{jk})I_k(\boldsymbol{\omega}_{jk})\}_{j=1}^{N_k}$. Since the original subsequence $\{n_j\} \subset \{n\}$ was arbitrary, we have the result $P\left(0 \in ch^o\{G_\theta(\boldsymbol{\omega}_{jn})I_k(\boldsymbol{\omega}_{jn})\}_{j=1}^{N}\right) \to 1$. □

To conclude, we present the proof of Proposition 4.1 from Section 4, which involves verifying Condition (C.5)' on prototypical examples of spectral estimating functions given in Section 3.2 (cf. Examples 1-3 there).

Proof of Proposition 4.1. First consider Example 1. Here $G_{j,\theta}(\boldsymbol{\omega}) = \cos\mathbf{h}_j'\boldsymbol{\omega} - \theta_j$, $1 \leq j \leq p$, where $\theta_j = corr(Z(\mathbf{h}_j), Z(\mathbf{0}))$. For any $1 \leq i, j \leq p$, $\left|\sum_{k=1}^N G_{i,\theta_0}(\boldsymbol{\omega}_{kn})G_{j,\theta_0}(\boldsymbol{\omega}_{kn})\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\right|$ is bounded above by a constant multiple of a sum of at most 16 terms of the form

$$V_{1n}(\mathbf{a},\mathbf{b}) \equiv \Big|\sum_{k=1}^{N} \exp(\iota[\mathbf{a}+\mathbf{b}]'\boldsymbol{\omega}_{kn})\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\Big|$$

for $\mathbf{a}, \mathbf{b} \in \{\mathbf{0}, \pm\mathbf{h}_i, \pm\mathbf{h}_j\}$. Using arguments similar to those in the proof of Lemma 7.1 and using (S.1), it is easy to show that

$$V_{1n}(\mathbf{a},\mathbf{b}) \leq C\prod_{k=1}^{d} g_n(\mathbf{e}_k'[\mathbf{t}+\mathbf{a}+\mathbf{b}])$$

where the constant C does not depend on $\mathbf{a}, \mathbf{b}$ (and n). Finally, using a change of variables for the case $a_1 = 1$, one can establish Condition (C.5)' (with the bound $O(\lambda_n^{\kappa d}(\log\lambda_n)^d) = o(\lambda_n b_n^2)$), as in the proof of Lemma 7.1.

Next consider Example 2. Here $\left|\sum_{k=1}^N G_{i,\theta_0}(\boldsymbol{\omega}_{kn})G_{j,\theta_0}(\boldsymbol{\omega}_{kn})\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\right|$ is bounded above by a constant multiple of a sum of at most 16 terms of the form

$$V_{2n}(\mathbf{a},\mathbf{b}) \equiv \Big|\sum_{k=1}^{N} \mathbb{1}_A(\boldsymbol{\omega}_{kn})\mathbb{1}_B(\boldsymbol{\omega}_{kn})\exp(\iota\mathbf{t}'\boldsymbol{\omega}_{kn})\Big|$$

where A and B are d-dimensional rectangles determined by $\mathbf{t}_1, \ldots, \mathbf{t}_p$ of

Example 2. As in (S.1), it is easy to show that for any $[a,b] \subset [-\infty,\infty]$,

$$\begin{aligned} & \Big| \sum_{-C^*\lambda_n^\eta \le j \le C^*\lambda_n^\eta} \mathbb{1}_{[a,b]}(j\lambda_n^{-\kappa}) \exp(\iota t j \lambda_n^{-\kappa}) \Big| \\ = & \Big| \sum_j \mathbb{1}\Big(-C^*\lambda_n^\eta \vee a\lambda_n^\kappa \le j \le C^*\lambda_n^\eta \wedge b\alpha_n^\kappa \Big) \exp(\iota t j \lambda_n^{-\kappa}) \Big| \le g_n(t), \end{aligned}$$

where $g_n(t) \equiv \begin{cases} (2C^*+1)\lambda_n^\eta & \text{if} \quad |(t\lambda_n^{-\kappa})_{2\pi}| \le \lambda_n^{-\eta} \\ \frac{C^*}{|(t\lambda_n^{-\kappa})_{2\pi}|} & \text{if} \quad \lambda_n^{-\eta} \le |(t\lambda_n^{-\kappa})_{2\pi}| \le \pi. \end{cases}$

Hence, Condition (C.5)' likewise follows for Example 2. The proof for Example 3 is similar to that for Example 1 and hence it is omitted. □

2. Additional Simulation results. To facilitate a direct comparison of different cases, we present the results from the simulation results for all three sets of lags, including the set $\mathbf{h}_1 = (1,1)', \mathbf{h}_2 = (1,-1)'$ reported in the main paper.

TABLE 1
Coverage percentage of 90% SFDEL regions for variogram model parameters θ (uniform design).

		$\lambda_n = 12$				$\lambda_n = 24$				$\lambda_n = 48$			
		$\mathbf{h}_1 = (1,1)', \mathbf{h}_2 = (1,-1)'$											
C	κ	100	400	900	1400	100	400	900	1400	100	400	900	1400
1	0.05	86.4	85.6	82.0	80.3	88.9	87.8	87.8	89.9	89.3	89.4	89.7	87.9
1	0.1	87.1	85.3	78.6	75.9	89.0	90.2	89.6	90.4	89.0	91.4	91.5	90.0
1	0.2	86.5	85.1	81.1	76.4	90.0	88.7	90.1	89.7	87.6	87.9	87.9	88.9
2	0.05	88.1	87.8	86.1	85.9	89.0	88.6	89.7	87.9	89.2	88.9	90.5	89.7
2	0.1	86.6	86.8	86.2	84.2	89.2	88.4	91.1	89.9	90.6	90.0	90.0	91.4
2	0.2	89.6	88.8	84.6	83.8	88.9	89.9	89.9	89.2	89.9	89.3	88.1	89.4
4	0.05	89.0	87.8	89.6	88.1	89.3	89.0	90.1	90.2	92.9	88.2	90.6	89.9
4	0.1	86.3	88.6	88.7	86.4	90.3	89.4	90.3	89.2	92.0	87.8	90.8	89.1
4	0.2	88.4	89.0	87.4	87.9	88.7	88.9	90.0	89.6	92.8	88.6	88.5	88.8
		$\mathbf{h}_1 = (1,1)', \mathbf{h}_2 = (3,3)'$											
C	κ	100	400	900	1400	100	400	900	1400	100	400	900	1400
1	0.05	88.4	87.9	81.8	82.8	89.7	90.6	88.6	90.8	89.6	91.3	90.0	90.8
1	0.1	89.0	87.1	83.3	80.2	88.8	89.7	89.0	89.5	89.7	89.5	89.9	88.9
1	0.2	88.5	86.1	83.7	81.1	90.2	90.5	89.5	89.7	89.1	89.7	89.4	89.0
2	0.05	89.1	88.9	88.2	86.5	89.4	89.5	90.4	91.0	88.7	88.9	89.7	89.3
2	0.1	90.0	88.9	89.2	87.2	87.7	90.3	91.3	89.9	89.3	89.6	89.1	91.0
2	0.2	89.3	89.8	86.3	85.9	89.3	90.5	88.6	88.6	88.8	89.8	87.3	89.0
4	0.05	90.7	89.4	90.9	89.2	89.6	89.5	89.5	89.8	93.3	89.9	90.8	88.6
4	0.1	88.7	89.5	90.9	89.1	90.0	88.9	90.2	89.6	92.6	87.7	89.4	88.8
4	0.2	89.4	89.8	90.6	90.0	89.7	88.4	89.2	90.8	93.5	88.5	89.9	90.1
		$\mathbf{h}_1 = (3,3)', \mathbf{h}_2 = (-3,-3)'$											
C	κ	100	400	900	1400	100	400	900	1400	100	400	900	1400
1	0.05	89.8	87.4	87.2	88.0	89.6	91.5	89.5	90.1	89.5	91.5	89.8	90.0
1	0.1	91.6	91.4	89.4	87.6	89.5	89.4	88.8	89.8	89.2	89.3	90.8	89.6
1	0.2	89.9	87.0	86.1	87.0	89.8	89.5	88.3	88.7	89.1	90.7	90.5	90.2
2	0.05	91.5	91.9	90.9	90.2	91.1	89.6	89.8	90.7	90.5	90.0	89.1	89.1
2	0.1	91.3	91.5	91.6	91.6	88.6	91.4	89.7	90.2	88.8	90.4	90.0	91.9
2	0.2	88.9	88.9	91.0	86.5	88.1	91.0	88.8	87.6	89.3	89.8	87.8	89.6
4	0.05	91.3	91.1	91.9	90.7	91.5	89.9	90.4	90.5	93.2	87.5	90.6	89.7
4	0.1	91.4	91.9	92.4	91.7	89.8	89.8	88.3	89.1	92.8	89.2	90.1	88.5
4	0.2	91.0	90.1	92.6	90.7	91.0	87.4	90.7	89.8	93.5	89.8	88.8	88.6

Table 2
Coverage percentage of 90% SFDEL regions for variogram model parameters θ *(non-uniform design)*

$\mathbf{h}_1 = (1,1)', \mathbf{h}_2 = (1,-1)'$

		$\lambda_n = 12$				$\lambda_n = 24$				$\lambda_n = 48$			
C	κ	100	400	900	1400	100	400	900	1400	100	400	900	1400
1	0.05	88.3	86.8	85.5	79.6	89.4	88.9	86.4	90.1	90.2	89.2	89.6	90.0
1	0.10	85.7	83.5	80.3	78.7	88.8	87.7	89.6	92.0	87.0	90.5	89.5	89.3
1	0.20	87.9	86.0	82.4	79.1	89.6	90.0	90.7	90.0	87.4	88.2	88.8	88.7
2	0.05	89.4	89.3	88.0	83.8	90.1	88.6	89.7	88.9	89.6	90.7	89.5	91.2
2	0.10	86.2	87.7	84.3	85.9	89.0	90.7	90.0	88.4	90.1	91.5	90.1	90.0
2	0.20	88.7	89.5	88.5	85.6	90.7	90.4	89.7	88.5	89.7	88.5	90.3	89.8
4	0.05	89.5	89.5	88.3	88.0	87.7	90.0	88.6	90.7	91.8	89.8	89.2	90.9
4	0.10	87.0	88.8	87.4	86.2	89.0	89.9	87.9	89.4	91.7	87.9	88.2	89.9
4	0.20	90.6	89.1	89.1	86.3	89.5	89.0	87.9	89.4	91.5	90.2	89.3	90.1

$\mathbf{h}_1 = (1,1)', \mathbf{h}_2 = (3,3)'$

C	κ	100	400	900	1400	100	400	900	1400	100	400	900	1400
1	0.05	90.4	88.7	86.1	83.0	88.8	89.2	87.6	90.3	90.1	89.4	89.9	89.0
1	0.10	87.3	84.7	81.3	81.5	88.7	88.4	90.2	90.6	88.8	92.2	90.6	90.2
1	0.20	88.8	87.0	85.8	83.4	89.7	90.5	90.5	89.9	87.6	89.9	89.3	88.7
2	0.05	91.0	90.0	88.4	86.8	90.5	90.0	89.5	89.6	90.2	89.6	91.0	92.4
2	0.10	89.5	90.8	87.6	88.1	88.7	90.4	89.7	90.4	89.4	91.4	90.8	88.8
2	0.20	87.8	90.1	89.4	86.3	89.6	90.5	89.9	89.4	90.6	87.8	88.2	90.4
4	0.05	89.2	90.2	89.3	90.8	89.3	91.6	89.8	90.4	93.9	90.4	89.2	88.4
4	0.10	88.3	88.6	89.4	89.0	89.6	91.4	89.3	90.1	92.6	89.4	89.4	89.7
4	0.20	91.1	90.2	89.9	89.0	89.6	88.3	88.9	89.9	91.3	89.8	90.4	88.2

$\mathbf{h}_1 = (3,3)', \mathbf{h}_2 = (-3,-3)'$

C	κ	100	400	900	1400	100	400	900	1400	100	400	900	1400
1	0.05	90.0	88.5	87.9	86.2	88.9	90.4	90.0	88.0	88.9	90.5	89.5	91.2
1	0.10	91.6	90.6	86.6	85.6	89.8	91.3	89.9	89.7	89.3	92.0	89.9	89.3
1	0.20	90.1	87.5	85.9	86.1	89.4	88.8	89.6	87.6	89.8	89.3	90.0	89.7
2	0.05	91.3	90.5	89.5	90.8	90.5	90.9	89.6	88.9	89.9	88.9	89.5	90.8
2	0.10	92.1	92.6	90.2	91.0	88.7	89.6	89.7	91.3	89.1	90.1	89.9	88.9
2	0.20	88.3	90.8	89.8	88.7	89.4	89.3	90.3	88.0	90.8	88.6	87.2	90.3
4	0.05	90.3	91.0	91.5	92.5	89.6	91.3	89.8	89.2	95.3	90.2	90.3	88.6
4	0.10	90.2	90.2	91.5	92.9	90.8	91.6	90.2	90.2	93.2	89.9	89.7	89.2
4	0.20	91.0	91.2	90.8	90.9	90.1	88.5	88.4	90.3	92.1	89.7	90.1	90.2

References.

[1] Doukhan, P. (1994). Mixing: properties and examples. In *Lecture Notes in Statistics*, **85** Springer-Verlag, New York.

[2] Lahiri, S. (2003). Central limit theorems for weighted sums of a spatial process under a class of stochastic and fixed designs. *Sankhya* **65** 356–388.

[3] Owen, A. B. (1990). Empirical likelihood ratio confidence regions. *Ann. Statist.* **18** 90–120.

Department of Mathematics
Lehigh University
Bethlehem, PA USA 18015
E-mail: sob210@lehigh.edu

Department of Statistics
North Carolina State University
Raleigh, NC USA 27695-8203
E-mail: snlahiri@ncsu.edu

Department of Statistics
Iowa State University
Ames, IA USA 50011
E-mail: dnordman@iastate.edu