Asymptotic Properties of Discrete Fourier Transforms for Spatial Data

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ABSTRACT

Asymptotic distribution of the Discrete Fourier Transformation (DFT) of spatial data under pure and mixed increasing- domain spatial asymptotic structures are studied under both deterministic and stochastic spatial sampling designs. The deterministic design is specified by a scaled version of the integer lattice in $\mathbb{R}^d$ while the data-sites under the stochastic spatial design are generated by a sequence of independent random vectors, with a possibly nonuniform density. A detailed account of the asymptotic joint distribution of the DFTs of the spatial data is given which, among other things, highlights the effects of the geometry of the sampling region and the spatial sampling density on the limit distribution. Further, it is shown that in both deterministic and stochastic design cases, for "asymptotically distant" frequencies, the DFTs are asymptotically independent, but this property may be destroyed if the frequencies are "asymptotically close". Some important implications of the main results are also given.

*Research partially supported by NSF grant no. DMS 0707139.

AMS (2000) subject classification: Primary 62M30, Secondary 62E20.

Keywords and phrases: Asymptotic independence, Central limit theorem, DFT, Random field, Spatial processes, Spatial design.

1 Introduction

In recent years, there has been a surge of research interest in the analysis of spatial data using the frequency domain approach; see for example, Hall and Patil (1994), Im et al. (2007), Fuentes (2002, 2005, 2007), and the references therein. At a heuristic level, the popularity of the frequency domain approach lies in the fact that for *equi-spaced* time series data, the discrete Fourier transform (DFT) of the observations are asymptotically independent (cf. Kawata (1966,1969), Fuller (1976) and Brockwell and Davis (1991), Lahiri (2003b)). As a result, it allows one to avoid accounting for the dependence in the data explicitly. However, validity of the asymptotic independence of the DFTs for spatial data remains largely unexplored. In contrast to the time series case where observations are usually taken at a regular interval of time and asymptotics is driven by the unidirectional flow of time, for random processes observed over space, several different types of spatial sampling designs and spatial asymptotic structures are relevant for practical applications. For example, image data are equi-spaced in the plane, but locations of the drilling-sites for mineral ores in a mine are usually irregularly spaced. Thus, the type of asymptotics that are appropriate in these applications are inherently different. In this paper, the asymptotic properties of the DFT for equi-spaced as well as irregularly spaced spatial data under different types of spatial asymptotic structures are investigated in detail.

For spatial data, there are two basic types of spatial asymptotic structures (cf. Cressie (1993)): (i) *pure increasing domain* (PID) and (ii) *infill*. When the neighboring data-sites remain separated by a minimum positive distance (in the limit) and the sampling region becomes unbounded with the sample size, one gets the PID asymptotic structure. This is the most common framework used for studying the large sample properties in the spatial case and may be considered as the spatial analogue of the asymptotic structure used in the time-series case. In contrast, when the sampling region remains bounded and the data-sites fill in the sampling region increasingly densely, one gets the *infill* asymptotic structure. This kind of asymptotic framework is mainly used in Mining and other Geostatistical applications. In some situations, a combination of these two frameworks, called the *mixed increasing domain* (MID) asymptotic structure is used (cf. Hall and Patil (1994)). Under MID asymptotics, the sampling region becomes unbounded and at the same time, the distances between the neighboring sampling sites tend to zero, as the sample size increases.

In this paper, the asymptotic joint distribution of a finite collection of DFTs of spatial data under the PID and MID asymptotic structures are investigated. It has been noted that the large sample behaviors of many standard inference procedures under the infill asymptotics are noticeably different from what can be obtained under the PID or MID asymptotic frameworks; See, for example, Cressie (1993), Lahiri (1996) , Loh (2005), Stein (1999), Ying (1993) and the references therein. Indeed, unlike the PID and MID cases, the asymptotic distributions of the DFTs under infill asymptotics are typically non-normal and the DFTs are typically asymptotically dependent for the general class of underlying spatial processes considered here. As a result, the case of pure infill asymptotics is not considered here and the results are proved only on the PID and MID asymptotic structures for regularly (gridded) and irregularly spaced spatial data.

To describe the main results of the paper, let $\{Z(\mathbf{s}) : \mathbf{s} \in I\!\!R^d\}$ be a zero mean stationary random field which is observed at finitely many locations $\mathbf{S}_N = \{\mathbf{s}_i : i = 1, \ldots, N\}$ in the sampling region $\mathcal{D} \subset I\!\!R^d$. It will be assumed that in the equi-spaced case, the data-sites $\{\mathbf{s}_i : i = 1, \ldots, N\}$ lie on a scaled version of the integer grid (call it $\mathcal{Z}^d$), while in the irregularly spaced spatial data case, the data-sites are generated by a stochastic sampling scheme. The Discrete Fourier Transform (DFT) of $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_N)\}$ is given by,

$$d_N(\boldsymbol{\omega}) = N^{-1/2} \sum_{j=1}^{N} Z(\mathbf{s}_j) \exp\left(\iota \boldsymbol{\omega}' \mathbf{s}_j\right), \quad \boldsymbol{\omega} \in I\!\!R^d, \tag{1.1}$$

where $\iota = \sqrt{-1}$ and B' denote the transpose of a matrix B. For $\boldsymbol{\omega} \in I\!\!R^d$, also define

$$\begin{aligned} C_N(\boldsymbol{\omega}) &= N^{-1/2} \sum_{j=1}^{N} \cos(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \\ S_N(\boldsymbol{\omega}) &= N^{-1/2} \sum_{j=1}^{N} \sin(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \end{aligned} \tag{1.2}$$

the cosine and the sine transforms of the data. Then, $d_N(\boldsymbol{\omega}) = C_N(\boldsymbol{\omega}) + \iota S_N(\boldsymbol{\omega})$. The main findings of our paper under both deterministic and stochastic sampling designs are:

(i) As in the time series case, under suitable regularity conditions, the asymptotic joint distributions of finite collections of the sine and cosine transforms are multivariate Gaussian under both deterministic and stochastic designs. However, in the stochastic design case, the asymptotic covariance critically depends on the spatial sampling density and the spatial asymptotic structure (PID *vs* MID); A complete description of their effects on the resulting limit distributions is given.

(ii) DFTs at unequal nonzero limiting frequencies are asymptotically independent.

(iii) In the fixed design case, for sampling regions of a *general shape* and for DFTs at ordinates converging to a *common* limiting frequency, asymptotic independence holds *if and only if* the ordinates are *asymptotically distant*. $\{\boldsymbol{\omega}_{jn}\}$ and $\{\boldsymbol{\omega}_{kn}\}$ are called asymptotically distant if $(\text{vol.}(\mathcal{D}))^{1/d} \|\boldsymbol{\omega}_{jn} - \boldsymbol{\omega}_{kn}\| \to \infty$ as $N \to \infty$. In the stochastic design case, similar result holds for a general sampling density. Thus, although the data-sites are irregularly spaced, the asymptotic behavior of the DFTs remains similar to that for regularly spaced spatial data. This is rather surprising and contrary to the folklore about lack of independence of DFTs for irregularly spaced time series data.

(iv) For two discrete Fourier frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ converging to the zero frequency, the corresponding sine and cosine transforms may exhibit different behavior depending on whether the frequency sequences approach zero at the same rate (*asymptotically symmetrically close case*) or at a different rate (*asymptotically close case*). See section 3.1.3 and section 4.3 for details.

(v) For sampling sites located on the d-dimensional integer grid, DFTs at all discrete Fourier frequencies are asymptotically independent when the sampling region is cubic. However, this is false for a sampling region of a general shape (including spheres, hyper-rectangles, etc.). Also for sampling sites on a scaled version of $\mathbb{Z}^d$ and a *rectangular sampling region*, asymptotic independence holds, provided the grid-increment in each direction is inversely proportional to the sides of the sampling region.

(vi) Under the stochastic design, for a *hyper-rectangular sampling region* and a *uniform* sampling density, asymptotic independence of DFTs holds even for asymptotically close frequency sequences. See Section 4 for more details.

Thus, in contrast to the time series case, the geometry of the sampling region plays an important role in determining the asymptotic independence of the DFTs of spatial data. The main tool used in the regular-grid case is a discrete version of the Riemann-Lebesgue Lemma (Cf. Section 6) that may be of some independent interest. For more details on the properties of the DFTs based on regularly spaced spatial data, see Section 3.

There are several important implications of the main results on asymptotic independence of the DFTs in the context of statistical inference for spatial data in the frequency domain, particularly under PID in the stochastic design case. For example, the usual formulation of the frequency domain bootstrap (FDB) (cf. Franke and Hardle (1992)), which makes use of the asymptotic independence of the full set of DFTs, may not work for spatial data when the sampling region is non-rectangular. Similarly, the popular nonparametric estimator of the covariance function of Hall and Patil (1994) for irregularly spaced spatial data may have a nontrivial bias under PID asymptotic structure and hence, will be inconsistent. see Section 5 for further discussion and details.

The rest of the paper is organized as follows. In Section 2, the theoretical framework for studying the asymptotic distributions of the DFTs for equi-spaced and irregularly spaced spatial data is introduced. In Section 3, the main results for the equi-spaced case under the PID and MID asymptotic structures are presented, while in Section 4, the results for the stochastic design case are stated. In Section 5, various implications of the main results are discussed in the context of frequency domain statistical inference for spatial data. Proofs of the two cases require qualitatively different arguments and are presented in Sections 6 and 7, respectively.

2 Theoretical Framework

Throughout the paper, a spatial asymptotic framework as in Lahiri (2003a) will be followed. Denote the variable driving the asymptotics by n. In Section 2.1, a formulation for the sampling region is given that is common to both deterministic and stochastic design cases. The descriptions of the two spatial designs for the regularly- and irregularly-spaced data-sites are next given in Sections 2.2 and 2.3, respectively. Regularity conditions on the random field $\{Z(\cdot)\}$ are given in Section 2.4.

2.1 Sampling Region

Let $\mathcal{D}_0$ be the prototype set for the sampling region $\mathcal{D} \equiv \mathcal{D}_n$, satisfying $\tilde{\mathcal{D}}_0 \subset \mathcal{D}_0 \subset \text{closure}(\tilde{\mathcal{D}}_0)$ for some open connected subset $\tilde{\mathcal{D}}_0$ of $(-1/2, 1/2]^d$ containing the origin. The sampling region $\{\mathcal{D}_n : n \geq 1\}$ is obtained by multiplying the prototype set $\mathcal{D}_0$ by λ_n, where $\{\lambda_n\}_{n\geq 1} \subset [1, \infty)$ is a sequence of real numbers such that $\lambda_n \uparrow \infty$ as $n \to \infty$, *i.e.*,

$$\mathcal{D}_n = \lambda_n \mathcal{D}_0.$$

It may be noted that under this formulation of the sampling region, sampling regions of a variety of shapes can be considered, such as polygonal, ellipsoidal, and star-shaped regions that can be non-convex. Also to avoid pathological cases, it is supposed that for any sequence of real numbers $\{b_n\}_{n\geq 1}$ such that $b_n \to 0+$ as $n \to \infty$, the number of cubes of the form $b_n(\mathbf{j} + [0,1)^d)$, $\mathbf{j} \in \mathbb{Z}^d$ that intersects both $\mathcal{D}_0$ and $\mathcal{D}_0^c$ is of the order $O([b_n]^{-(d-1)})$ as $n \to \infty$. This boundary condition holds for most regions of practical interest.

2.2 Sampling design for regularly-spaced data-sites

To describe the deterministic design case, let Δ be a $d \times d$ diagonal matrix with finite positive diagonal elements $\delta_k, k = 1, \cdots, d$ and let $\mathcal{Z}^d = \{\Delta \mathbf{i} : \mathbf{i} \in \mathbb{Z}^d\}$. Thus, the lattice $\mathcal{Z}^d$ has an increment δ_k in the kth direction, $k = 1, \cdots, d$. For the PID, it is assumed that the random process $Z(\mathbf{s})$ is observed at the sampling sites $\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\}$ defined by

$$\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\} = \{\mathbf{s} \in \mathcal{Z}^d : \mathbf{s} \in \mathcal{D}_n\} = \mathcal{D}_n \cap \mathcal{Z}^d.$$

Note that under the PID, the sampling sites are separated by a minimum distance $\delta_0 \equiv \min\{\delta_k : k = 1, \ldots, d\}$ for all n, the sampling region $\mathcal{D}_n$ grows to $\mathbb{R}^d$ as $n \to \infty$ and the sample size N_n satisfies the relation

$$N_n \sim \text{ vol.}[\Delta^{-1}\mathcal{D}_0]\lambda_n^d, \tag{2.1}$$

where vol.$[A]$ denotes the volume (*i.e.,* the Lebesgue measure) of a set A in $\mathbb{R}^d$ and for two positive sequences $\{s_n\}$ and $\{t_n\}$ let us write, $s_n \sim t_n$ if $\lim_{n\to\infty} s_n/t_n = 1$.

Next the MID structure in the fixed design case is described. Let $\{\eta_n\}_{n\geq 1}$ be a sequence of non-increasing positive real numbers such that $\eta_n \downarrow 0$ as $n \to \infty$. It is supposed that the random process $Z(\mathbf{s})$ is observed at the sampling sites $\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\}$, defined by

$$\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\} = \{\mathbf{s} \in \eta_n \mathcal{Z}^d : \mathbf{s} \in \mathcal{D}_n\} = \mathcal{D}_n \cap \eta_n \mathcal{Z}^d.$$

Thus, the data-sites $\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\}$ are given by the points on the scaled lattice $\eta_n \mathcal{Z}^d$ that lie in the sampling region $\mathcal{D}_n$. Under MID, the lattice $\eta_n \mathcal{Z}^d$ becomes finer as $n \to \infty$ and thus, fills any given region of $I\!R^d$ (and hence, of $\mathcal{D}_n$) with an increasing density. Note that in the MID case,

$$N_n \sim \text{ vol.}[\Delta^{-1}\mathcal{D}_0]\lambda_n^d \eta^{-d}, \qquad (2.2)$$

implying that under the MID structure, the sample size N_n is of a *larger order* of magnitude than the volume of $\mathcal{D}_n$, given by vol.$[\Delta^{-1}\mathcal{D}_0]\lambda_n^d$.

2.3 A stochastic sampling design for the irregularly spaced case

Let $f(\mathbf{x})$ be a continuous probability density function on $\mathcal{D}_0$ such that the support of $f(\cdot)$ is the closure of $\mathcal{D}_0$. Let $\{\mathbf{X}_k\}_{k\geq 1}$ be a sequence of independent and identically distributed (iid) random vectors with probability density $f(\mathbf{x})$. In the stochastic design case, for simplicity of notation, denote the sample size by n (Note that in the fixed design case, sample size equals the size of $\mathcal{D}_n \cap \mathcal{Z}^d$ which need not be equal to n for a given prototype set $\mathcal{D}_0$ and for a given sequence $\{\lambda_n\}$, leading to the notation N_n. But, due to the absence of a regular grid structure, this problem does not appear in the stochastic design case and one may simply use n to denote the sample size). In the stochastic design case, the sampling sites $\mathbf{s}_i$'s are obtained by the following relation

$$\mathbf{s}_i \;\; \equiv \;\; \mathbf{s}_{in} \; = \; \lambda_n \mathbf{x}_i, \;\; 1 \leq i \leq n.$$

This formulation improves upon the standard approach to modeling irregularly spaced sampling sites using a homogeneous Poisson point process. For such a process, the expected number of points in a region is proportional to the volume of the region and given the total number of points in any region, the points are independent and form a random sample from the uniform distribution over the region. However, the formulation here allows the number of sampling sites to grow at a different rate than the volume of the sampling region and also allows the sampling sites to have a *non-uniform density* over the sampling region.

In the stochastic design case, the concepts of the PID and the MID structures are determined by the relative growth rates of the sample size n and the volume of the sampling region $\mathcal{D}_n$ (cf. Cressie (1993), Lahiri (2003a)). When $n/\lambda_n^d \to c_*$ for some finite positive constant c_*, it is regarded as the PID asymptotic structure (cf. (2.1)) under the stochastic design. On the other hand, if $n/\lambda_n^d \to \infty$ as $n \to \infty$, it corresponds to the MID case (cf. (2.2)).

2.4 Regularity conditions on the random field

Let $I\!N = \{1, 2, \ldots\}$ denote the set of all positive integers. For $p \in I\!N$, let I_p denote the identity matrix of order p. In addition to the standard ℓ^2-distance $\|\cdot\|$, let $\|\cdot\|_1$ denote the ℓ^1 distance on $I\!R^d$. Let $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ be a weakly dependent random field with an integrable autocovariance function $\rho(\mathbf{s}) = \mathrm{cov}(Z(\mathbf{s}), Z(\mathbf{0}))$. Then, the $Z(\cdot)$-process has a spectral density ψ on $I\!R^d$ satisfying

$$\rho(\mathbf{s}) = \int \exp(\iota \mathbf{s}'\boldsymbol{\omega})\psi(\boldsymbol{\omega})d\boldsymbol{\omega}, \quad \mathbf{s} \in I\!R^d. \tag{2.3}$$

Suppose that the random field $Z(\cdot)$ satisfies a spatial version of the strong mixing condition, which is defined as follows: For $E_1, E_2 \subset I\!R^d$, let

$$\alpha_1(E_1, E_2) = \sup_{A_i \in \sigma_Z(E_i), i=1,2} |P(A_1 \cap A_2) - P(A_1)P(A_2)|,$$

where $\sigma_Z(E)$ denotes the σ-algebra generated by the random variables $\{Z(\mathbf{s}) : \mathbf{s} \in E\}$. Let $\delta(E_1, E_2) = \inf\{\|\mathbf{x} - \mathbf{s}\|_1 : \mathbf{x} \in E_1, \mathbf{s} \in E_2\}$. For $a > 0, b > 0$, the mixing coefficient of the random field $\{Z(\cdot)\}$ is defined as

$$\alpha(a; b) = \sup\{\alpha_1(E_1, E_2) : E_i \in \mathbb{C}_b,\ i = 1, 2,\ \delta(E_1, E_2) \geq a\},$$

where $\mathbb{C}_b$ is the collection of d-dimensional sets with volume b or less. Note that in the definition above, the sets E_1, E_2 are of finite volumes. For $d \geq 2$, this is important (cf. Bradley, 1989, 1993); unbounded E_1's and E_2's in the definition of the strong mixing coefficient makes the random field ρ-mixing (which is a *smaller* class). For simplicity of exposition, further assume that

$$\alpha(a, b) \leq \gamma_1(a)\gamma_2(b), \quad a, b \in (0, \infty), \tag{2.4}$$

where, without loss of generality (w.l.g.), $\gamma_1(\cdot)$ is a left continuous, non increasing function satisfying $\lim_{m\to\infty} \gamma_1(m) = 0$ and $\gamma_2(\cdot)$ is a right continuous, non decreasing function that is bounded for $d = 1$ (but it may be unbounded for $d > 1$) (cf. Lahiri (2003a)).

The following regularity conditions will be used to prove the results.

ASSUMPTIONS

(A.1) There exists a $\tau \in (0, \infty)$ such that $E|Z(\mathbf{s})|^{2+\tau} < \infty$ and $\int_1^\infty a^{d-1}\gamma_1(a)^{\frac{\tau}{2+\tau}} da < \infty$ for some $\tau > 0$.

(A.2) For $d \geq 2$, $\gamma_2(b) = o(b^\kappa)$ as $b \to \infty$, where, with τ is as in (A.1), $\kappa = \frac{2}{3(\tau-1)}$ for $\tau \in (2, \infty)$ and $\kappa = 2/3$ for $\tau \in (0, 2]$.

3 Results in the regularly-spaced case

3.1 Results under PID

3.1.1 Definition of the DFTs

For the PID asymptotic structure in the equi-spaced case, the discrete Fourier transform (DFT) of $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_{N_n})\}$ is given by,

$$\begin{aligned} d_n^P(\boldsymbol{\omega}) &\equiv N_n^{-1/2} \sum_{i=1}^{N_n} Z(\mathbf{s}_i) \exp\left(\iota \boldsymbol{\omega}' \mathbf{s}_i\right) \\ &= N_n^{-1/2} \sum_{\mathbf{j} \in J_n} Z(\Delta \mathbf{j}) \exp\left(\iota \boldsymbol{\omega}' \Delta \mathbf{j}\right), \end{aligned} \tag{3.1}$$

where $J_n = \{\mathbf{j} \in \mathbb{Z}^d : \Delta \mathbf{j} \in \mathcal{D}_n\}$ and where recall that $\iota = \sqrt{-1}$ and B' denote the transpose of a matrix B. Similarly, for $\boldsymbol{\omega} \in I\!R^d$, let us define

$$\begin{aligned} C_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j} \in J_n} Z(\Delta \mathbf{j}) \cos\left(\boldsymbol{\omega}' \Delta \mathbf{j}\right), \\ S_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j} \in J_n} Z(\Delta \mathbf{j}) \sin\left(\boldsymbol{\omega}' \Delta \mathbf{j}\right), \end{aligned} \tag{3.2}$$

the cosine and the sine transforms of the data. Then, $d_n^P(\boldsymbol{\omega}) = C_n^P(\boldsymbol{\omega}) + \iota S_n^P(\boldsymbol{\omega})$. Under the fixed design case with the PID asymptotic structure, the process $Z(\cdot)$ is observed at regularly spaced locations on the grid $\mathcal{Z}^d$. In such a case, the spectrum of the observations is concentrated within the frequency band

$$\Pi_\Delta \equiv \Delta^{-1}(-\pi, \pi]^d = (-\pi\delta_1^{-1}, \pi\delta_1^{-1}] \times \ldots \times (-\pi\delta_d^{-1}, \pi\delta_d^{-1}].$$

The whole frequency space $I\!R^d$ is partitioned into (hyper-)rectangles of volume $(2\pi)^d \prod_{i=1}^d \delta_i^{-1}$, and the spectrum at a given point in the "principal band" Π_Δ is obtained by superimposing the spectra at congruent points from the partition $\{(\frac{i_1\pi}{\delta_1} \pm \frac{\pi}{\delta_1}] \times \ldots \times (\frac{i_1\pi}{\delta_d} \pm \frac{\pi}{\delta_d}] : (i_1, \ldots, i_d)' \in \mathbb{Z}^d\}$. Thus, it is easy to check that the spectral density ψ_Δ (say) of the $Z(\cdot)$-process on the lattice $\mathcal{Z}^d$ can be expressed in terms of the spectral density ψ of the continuous stationary process $Z(\cdot)$ as

$$\psi_\Delta(\boldsymbol{\omega}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \psi\left(\boldsymbol{\omega} + 2\pi\Delta^{-1}\mathbf{k}\right), \ \boldsymbol{\omega} \in \Pi_\Delta. \tag{3.3}$$

It turns out that the asymptotic distribution of the DFTs in the fixed design PID case depends on the spatial dependence structure of the random field $Z(\cdot)$ only through ψ_Δ.

3.1.2 Asymptotic distribution at nonzero frequencies

Consider the asymptotic joint distribution of $(d_n^P(\boldsymbol{\omega}_{1n}), \cdots, d_n^P(\boldsymbol{\omega}_{rn}))$ for a finite collection of frequencies $\boldsymbol{\omega}_{1n}, \ldots, \boldsymbol{\omega}_{rn}$, $1 \leq r < \infty$. In analogy to the equi-spaced observations in the time series case, here it is supposed that the $\boldsymbol{\omega}_{jn}$'s are of the form

$$\boldsymbol{\omega}_{jn} = 2\pi\lambda_n^{-1}\Delta^{-1}\mathbf{k}_{jn},\ \mathbf{k}_{jn} \in \mathbb{Z}^d, \quad \text{and} \quad \boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j \in \Pi_\Delta \quad \text{as} \quad n \to \infty. \tag{3.4}$$

The first result concerns the asymptotic joint distribution of the cosine and sine transforms at $\boldsymbol{\omega}_{1n}, \ldots, \boldsymbol{\omega}_{rn}$ in the case where $\pm\boldsymbol{\omega}_j$'s are distinct and nonzero elements of Π_Δ^0, where $\Pi_\Delta^0 = (-\pi\delta_1^{-1}, \pi\delta_1^{-1}) \times \ldots \times (-\pi\delta_d^{-1}, \pi\delta_d^{-1})$, is the interior of Π_Δ.

Theorem 3.1. *Suppose that, $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ is a zero mean stationary random field satisfying assumptions (A.1) and (A.2). Also suppose that for $j = 1, \ldots, r$, $r \in \mathbb{N}$, $\{\boldsymbol{\omega}_{jn}\}$ are sequences of the form (3.4) such that $\boldsymbol{\omega}_j \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}$ and $\boldsymbol{\omega}_j \pm \boldsymbol{\omega}_k \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}$ for all $1 \le j \ne k \le r$. Then, $[C_n^P(\boldsymbol{\omega}_{1n}), S_n^P(\boldsymbol{\omega}_{1n}), \cdots, C_n^P(\boldsymbol{\omega}_{rn}), S_n^P(\boldsymbol{\omega}_{rn})] \xrightarrow{d} N[\mathbf{0}, \Sigma]$, where Σ is a block diagonal matrix with r blocks of the form $A_j I_2$, where, with $c_0 = \frac{1}{2}(2\pi)^d/[\prod_{i=1}^d \delta_i]$, $A_j = c_0\psi_\Delta(\boldsymbol{\omega}_j)$ for $j = 1, \cdots, r$.*

Theorem 3.1 implies that for each single sequence $\{\boldsymbol{\omega}_{jn}\}$ converging to a non-zero frequency $\boldsymbol{\omega}_j \in \Pi_\Delta^0$, the corresponding sine and cosine transforms are asymptotically independent. Further, since the covariance matrix of the limiting Gaussian distribution is diagonal, any collection of *disjoint* subsets of the $2r$ cosine and sine transforms are also asymptotically independent. In particular, under the conditions of the theorem, the DFTs $(d_n^P(\boldsymbol{\omega}_{1n}), \cdots, d_n^P(\boldsymbol{\omega}_{rn}))$ are asymptotically independent and their asymptotic distribution depends on the dependence structure of the spatial process $\{Z(\cdot)\}$ only through the folded spectral density $\psi_\Delta(\cdot)$.

Next consider the case when the limit frequencies are not necessarily distinct. In this case, the joint asymptotic normality continues to hold under the regularity conditions of Theorem 3.1 on the random field $\{Z(\cdot)\}$. However, the asymptotic independence of the DFTs may no longer hold. To state the main results in a transparent manner, attention is restricted to the case $r = 2$ with a common nonzero limit frequency, although the conclusions do generalize to the case $r > 2$ in an obvious manner. Accordingly, consider the asymptotic joint distribution of $(d_n^P(\boldsymbol{\omega}_{1n}), d_n^P(\boldsymbol{\omega}_{2n}))'$, with $\boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j$ for $j = 1, 2$, where $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega} \ne \mathbf{0}$. Let $\boldsymbol{\omega}_{1n}^{(p)}$ and $\boldsymbol{\omega}_{2n}^{(p)}$ denote the p-th ordinate of the respective vectors $\boldsymbol{\omega}_{1n}$ and $\boldsymbol{\omega}_{2n}$, $p = 1, \cdots, d$. The limit behavior of the DFTs can be different depending on the closeness of the sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$, as specified below:

Definition: (i) $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are called *asymptotically distant* if

$$|\lambda_n(\omega_{1n}^{(p)} - \omega_{2n}^{(p)})| \to \infty \quad \text{as} \quad n \to \infty \quad \text{for at least one} \quad p = 1, \ldots, d. \tag{3.5}$$

(ii) $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are called *asymptotically close* if

$$\left.\begin{array}{l} \lambda_n(\omega_{1n}^{(p)} - \omega_{2n}^{(p)}) \to 2\pi\Delta^{-1}\ell_p \quad \text{as} \quad n \to \infty \quad \text{for all} \ \ p = 1, \cdots, d \\ \quad \text{with} \quad \boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)' \in \mathbb{Z}^d \quad \text{and} \quad \sum_{p=1}^d |\ell_p| \ne 0. \end{array}\right\} \tag{3.6}$$

Then the following result holds:

Theorem 3.2. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1) and (A.2). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are two sequences satisfying (3.4) with $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega}$ where $\boldsymbol{\omega} \neq \mathbf{0}$ and $2\boldsymbol{\omega} \in \Pi^0_\Delta$.*

(a) (Asymptotically distant frequencies): Under (3.5),

$$[C_n^P(\boldsymbol{\omega}_{1n}), S_n^P(\boldsymbol{\omega}_{1n}), C_n^P(\boldsymbol{\omega}_{2n}), S_n^P(\boldsymbol{\omega}_{2n})]' \xrightarrow{d} N[\mathbf{0}, \Sigma], \tag{3.7}$$

where $\Sigma = c_0 \psi_\Delta(\boldsymbol{\omega}) I_4$, with $c_0 = 2^{-1}[(2\pi)^d / \prod_{i=1}^d \delta_i]$.

(b) (Asymptotically close frequencies): Under (3.6), (3.7) holds with

$$\Sigma = \begin{bmatrix} A_1 & 0 & A_2 & A_3 \\ & A_1 & -A_3 & A_2 \\ & & A_1 & 0 \\ & & & A_1 \end{bmatrix}, \tag{3.8}$$

where $A_1 = c_0 \psi_\Delta(\boldsymbol{\omega})$, $A_2 = c_0 \psi_\Delta(\boldsymbol{\omega}) \phi_1(2\pi \boldsymbol{\ell})$, and $A_3 = c_0 \psi_\Delta(\boldsymbol{\omega}) \phi_2(2\pi \boldsymbol{\ell})$. Here, $\phi_1(\cdot)$ and $\phi_2(\cdot)$ respectively denote the real and the imaginary parts of the characteristic function of the uniform distribution on $\Delta^{-1}\mathcal{D}_0$.

Theorem 3.2 shows that for any two *asymptotically distant* sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ of frequencies, all four sine and cosine transforms are asymptotically independent and hence, the corresponding DFT's are asymptotically independent. However, for *asymptotically close* frequencies in the neighborhood of a nonzero frequency $\boldsymbol{\omega}$, this may no longer be true. Theorem 3.2 reveals some interesting behavior of the sine and the cosine transforms in this case. From the form of the asymptotic covariance matrix, it is clear that the sine and the cosine transforms *along a given* sequence of ordinates $\boldsymbol{\omega}_{jn}$ (with a fixed $j \in \{1, 2\}$) are asymptotically independent, but any combination of sine and cosine transforms corresponding to *different* ordinates (say, one at $\boldsymbol{\omega}_{jn}$ and the other at $\boldsymbol{\omega}_{kn}$ for $j \neq k$) may have a *non-zero correlation* in the limit. Note that if $\Delta^{-1}\mathcal{D}_0$ is symmetric around zero (in the sense $\mathbf{x} \in \Delta^{-1}\mathcal{D}_0$ implies $-\mathbf{x} \in \Delta^{-1}\mathcal{D}_0$), then the function $\phi_2(\cdot)$ is identically zero and the cross-correlation between the sine and cosine transforms vanish ($A_3 = 0$). However, for sampling regions of a *general* shape, the correlation between the two cosine transforms need not vanish, and therefore, the DFTs along $\boldsymbol{\omega}_{1n}$ and $\boldsymbol{\omega}_{2n}$ are typically not asymptotically independent.

Next consider the important *special case, where $\Delta^{-1}\mathcal{D}_0$ is an integer multiple of $(-\frac{1}{2}, \frac{1}{2}]^d$*. In this case, both $A_2 = 0$ and $A_3 = 0$, and therefore, the cross-correlation between the sine and the cosine transforms vanish in the limit. As a result, the corresponding DFTs are asymptotically independent. Some instances of this special case are:

(i) $\delta_i = 1$ for all $i = 1, \ldots, d$, and $\mathcal{D}_0 = (-1/2, 1/2]^d$.

(ii) The prototype set $\mathcal{D}_0 = (-a_1, a_1] \times \ldots \times (-a_d, a_d]$ for some $a_1, \ldots, a_d \in (0, 1/2)$ and $\delta_i = a_i^{-1}$ for all $i = 1, \ldots, d$.

Under (i), the spatial sampling sites lie on the integer grid $\mathbb{Z}^d$ and the sampling region is a (hyper-)cube in $I\!\!R^d$. Here, the DFTs are asymptotically independent even when the frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are asymptotically close. The behavior of the DFTs in this case is analogous to that for time series data observed over equispaced time points. However, it turns out that the asymptotic independence of the DFTs at asymptotically close frequencies can also hold for non-cubic regions and for non-equispaced spatial data-sites. Instance (ii) above corresponds to a hyper-rectangular sampling region. Here, the same conclusions as in Instance (i) are possible, if for each $i = 1, \ldots, d$, the grid increment δ_i in the ith direction is set as the inverse of the length of the rectangular prototype set in the that direction.

To summarize the main implications of Theorem 3.2, DFTs at asymptotically distant frequencies are asymptotically independent, and the asymptotic independence of DFTs may also hold for asymptotically close frequencies in certain *special* cases. However, for a non-rectangular sampling region, asymptotic independence of DFTs at asymptotically close ordinates typically fails in the spatial case, even for regularly spaced sampling sites. Thus, the shape of the sampling region and the sampling grid plays an important role in determining the behavior of the DFTs in the spatial case, which sets it apart from the familiar weakly dependent time series set up.

Remark: There is a dual to Theorem 3.2, where $\boldsymbol{\omega}_1 = -\boldsymbol{\omega}_2 = \boldsymbol{\omega}$ and $2\boldsymbol{\omega} \in \Pi^0_\Delta \setminus \{\mathbf{0}\}$. In this case, conclusions similar to part (a) of Theorem 3.2 hold if $\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\| \to \infty$. For $\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n}) \to 2\pi\Delta^{-1}\boldsymbol{\ell}$ for some $\boldsymbol{\ell} \in Z\!\!\!Z^d \setminus \{\mathbf{0}\}$, the asymptotic normality of the cosine and sine transforms as in (3.7) continues to hold under (A.1) and (A.2), but with the following limiting covariance matrix:

$$\Sigma = \begin{bmatrix} A_1 & 0 & A_2 & A_3 \\ & A_1 & A_3 & -A_2 \\ & & A_1 & 0 \\ & & & A_1 \end{bmatrix}, \tag{3.9}$$

where A_1, A_2, A_3 are as in part (b) of Theorem 3.2. In particular, for the special cases (i) and (ii) of cubic and rectangular sampling regions considered above, the asymptotic independence of the DFTs holds even for the asymptotically close frequencies, but not necessarily for sampling regions of a general shape.

3.1.3 Asymptotic distribution for the zero limiting frequency

Next consider the case where $\boldsymbol{\omega}_{jn}$'s converge to the zero frequency. Here, some extra care must be taken while studying the asymptotic behavior of the DFTs due to the special role played by the zero frequency in the definitions of the sine and cosine transforms. For the zero frequency limit, suppose that the discrete Fourier ordinates $\boldsymbol{\omega}_n = 2\pi\lambda_n^{-1}\Delta^{-1}\mathbf{k}_n$ satisfy the following regularity condition:

$$\boldsymbol{\omega}_n = 2\pi\lambda_n^{-1}\Delta^{-1}\mathbf{k}_n, \ \mathbf{k}_n \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \quad \text{and} \quad \boldsymbol{\omega}_n \to \mathbf{0} \quad \text{as} \quad n \to \infty. \tag{3.10}$$

The choice $\mathbf{k}_n = \mathbf{0}$ in (3.10) is ruled out in order to avoid some nonstandard asymptotic behavior of the DFTs. To appreciate why, suppose that $\mathbf{k}_n = \mathbf{0}$ along a subsequence and $\mathbf{k}_n \neq \mathbf{0}$ along a different subsequence, but $\boldsymbol{\omega}_n \to \mathbf{0}$. Then the sine transform is (identically) equal to zero along the first subsequence, but it has a non-degenerate limit distribution along the other subsequence, thereby destroying the convergence of the full sequence.

As before, for clarity of exposition, attention is restricted to the asymptotic distribution of the DFTs along two sequences of frequencies $\{\boldsymbol{\omega}_{jn}\}$, $j = 1, 2$ satisfying (3.10). The case of an arbitrary finite number of such frequency sequences can be handled in a straightforward manner, with added notational complexity. In comparison to a non-zero limit frequency, three situations arising from the relative orders of magnitude of the sequences $\{\boldsymbol{\omega}_{jn}\}$, $j = 1, 2$ should be considered:

(i) $\|\lambda_n(\boldsymbol{\omega}_{1n} \pm \boldsymbol{\omega}_{2n})\| \to \infty$

(ii) Exactly one of the sequences $\{\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\|\}$ and $\{\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\|\}$ tends to infinity and the other has a finite limit.

(iii) Both sequences $\{\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\|\}$ and $\{\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\|\}$ have finite limits.

The first two cases are the analogs of the "asymptotically distant" and "asymptotically close" cases considered in the last section. However, the situation covered in the third part can occur *only* for the zero limit frequency case, as both $\boldsymbol{\omega}_{1n}$ and $-\boldsymbol{\omega}_{1n}$ can be close to $\boldsymbol{\omega}_{2n}$ *simultaneously*. This will be referred to as the "asymptotically symmetrically close" case. The asymptotic behaviors of the cosine- and sine-transforms of spatial data under PID in these three cases are given by the following theorem.

Theorem 3.3. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ is a zero mean stationary random field satisfying (A.1) and (A.2). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are two sequences satisfying (3.10).*

(a) If $\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\| \to \infty$ and $\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\| \to \infty$, then

$$[C_n^P(\boldsymbol{\omega}_{1n}), S_n^P(\boldsymbol{\omega}_{1n}), C_n^P(\boldsymbol{\omega}_{2n}), S_n^P(\boldsymbol{\omega}_{2n})]' \xrightarrow{d} N\left[\mathbf{0}, \Sigma^{(a)}\right], \tag{3.11}$$

where $\Sigma^{(a)} = c_0 \psi_\Delta(\mathbf{0}) I_4$.

(b) Suppose that $\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\| \to \infty$ but $\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in \mathbb{R}^d$. Then, (3.11) holds with $\Sigma^{(a)}$ replaced by $\Sigma^{(b)}$, where $\Sigma^{(b)}$ has the form as in (3.8) with A_i's are replaced with $A_i^{[0]}$'s, for $i = 1, 2, 3$ where $A_1^{[0]} = c_0 \psi_\Delta(\mathbf{0})$, $A_2^{[0]} = c_0 \psi_\Delta(\mathbf{0}) \phi_1(\Delta \mathbf{z}_{12})$, and $A_3^{[0]} = c_0 \psi_\Delta(\mathbf{0}) \phi_2(\Delta \mathbf{z}_{12})$.

(c) Suppose that $\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n}) \to \mathbf{y}_{12} \in \mathbb{R}^d$ and $\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in \mathbb{R}^d$. Then, (3.11) holds with $\Sigma^{(a)}$ replaced by $\Sigma^{(c)}$, where

$$\Sigma^{(c)} = \sigma_\infty^2 \begin{bmatrix} [1+\tilde\phi_1(2\mathbf{y}_1)] & \tilde\phi_2(2\mathbf{y}_1) & [\tilde\phi_1(\mathbf{y}_{12})+\tilde\phi_1(\mathbf{z}_{12})] & [\tilde\phi_1(\mathbf{y}_{12})+\tilde\phi_1(\mathbf{z}_{12})] \\ & [1-\tilde\phi_1(2\mathbf{y}_1)] & [\tilde\phi_2(\mathbf{y}_{12})-\tilde\phi_2(\mathbf{z}_{12})] & [\tilde\phi_1(\mathbf{z}_{12})-\tilde\phi_2(\mathbf{y}_{12})] \\ & & [1+\tilde\phi_1(2\mathbf{y}_2)] & \tilde\phi_2(2\mathbf{y}_2) \\ & & & [1-\tilde\phi_1(2\mathbf{y}_2)] \end{bmatrix}$$

with $\sigma_\infty^2 = c_0\psi_\Delta(\mathbf{0})$, $\mathbf{y}_1 = (\mathbf{y}_{12}+\mathbf{z}_{12})/2$, $\mathbf{y}_2 = (\mathbf{y}_{12}-\mathbf{z}_{12})/2$, *and* $\tilde\phi_j(\boldsymbol{\omega}) = \phi_j(\Delta\boldsymbol{\omega})$, $\boldsymbol{\omega} \in I\!R^d$, $j = 1, 2$.

Thus, from Theorem 3.3, it follows that the sine and the cosine transforms at *non-zero ordinates converging to the zero frequency* have very similar asymptotic behavior as in the case of a non-zero limit frequency for the "asymptotically distant" and "asymptotically close" parts (cf. parts (a) and (b) of Theorems 3.2 and 3.3). In particular, asymptotic independence of the DFTs continues to hold for "asymptotically distant" discrete Fourier ordinates converging to zero. For "asymptotically close" ordinates converging to zero, DFTs are typically asymptotically dependent; For such sequences of ordinates, asymptotic independence of the DFTs holds in the special case where $\Delta^{-1}\mathcal{D}_0$ is an integer multiple of the d-cube $(-1/2, 1/2]^d$, as noted in the discussion of Theorem 3.2 above. Finally, for the "asymptotically symmetrically close" ordinates, it is clear that *every* possible pairs of sine and cosine transforms may have nontrivial asymptotic correlations for sampling regions of a general shape and hence, the DFTs typically are not asymptotically independent.

Next consider the special case where $\Delta^{-1}\mathcal{D}_0$ is d-cubic. Note that, in this case, $\tilde\phi_k(\mathbf{y}_j) = \phi_k(2\pi\boldsymbol{\ell}_j)$ for some $\boldsymbol{\ell}_j \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, for all $j, k \in \{1, 2\}$ and thus, all off-diagonal terms in $\Sigma^{(c)}$ vanish. In this special case, asymptotic independence of the DFTs hold even for 'asymptotically symmetrically close" ordinates converging to the zero frequency, as in the time series case. But, for sampling regions of a general shape, the DFTs are typically dependent in the limit in the 'asymptotically close' and 'asymptotically symmetrically close' cases.

Remark As in the last section, there is a dual to part (b) of Theorem 3.3. For $\|\lambda_n(\boldsymbol{\omega}_{1n}-\boldsymbol{\omega}_{2n})\| \to \infty$ but $\lambda_n(\boldsymbol{\omega}_{1n}+\boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in I\!R^d$, asymptotic normality of the cosine- and sine-transforms holds where the limiting covariance matrix is given by (3.9) with A_i's replaced by $A_i^{[0]}$'s, $i = 1, 2, 3$.

Remark For completeness, consider the case where $\boldsymbol{\omega}_n = \mathbf{0}$ for all $n \geq 1$. In the case, $S_n(\boldsymbol{\omega}_n) = 0$ for all $n \geq 1$, while $C_n(\boldsymbol{\omega}_n) = N_n^{-1/2}\sum_{i=1}^{N_n} Z(\mathbf{s}_i)$ and $C_n(\boldsymbol{\omega}_n) \to^d N(0, 2c_0\psi_\Delta(\mathbf{0}))$ solely under Assumption (A.1) and (A.2) (cf. Theorem 4.3, Lahiri (2003a))

3.1.4 Results for mean-corrected DFTs

In many applications, the random field $\{Z(\mathbf{s})\}$ has a mean $\mu = EZ(\mathbf{0})$ that is unknown. In such situations, the DFT defined in Sections 3.1.1 is often replaced by its mean corrected version:

$$\tilde d_n^P(\boldsymbol{\omega}) \equiv N_n^{-1/2} \sum_{\mathbf{j}\in J_n} [Z(\Delta\mathbf{j}) - \bar Z_n] \exp\left(\iota\boldsymbol{\omega}'\Delta\mathbf{j}\right) \tag{3.12}$$

where, as before, $J_n = \{\mathbf{j} \in \mathbb{Z}^d : \Delta\mathbf{j} \in \mathcal{D}_n\}$ and $\bar{Z}_n = N_n^{-1} \sum_{\mathbf{j}\in J_n} Z(\Delta\mathbf{j})$ is the sample mean. Similarly, let us define

$$\begin{aligned} \tilde{C}_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j}\in J_n} [Z(\Delta\mathbf{j}) - \bar{Z}_n]\cos\left(\boldsymbol{\omega}'\Delta\mathbf{j}\right), \\ \tilde{S}_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j}\in J_n} [Z(\Delta\mathbf{j}) - \bar{Z}_n]\sin\left(\boldsymbol{\omega}'\Delta\mathbf{j}\right), \end{aligned} \tag{3.13}$$

the mean corrected versions of the cosine and the sine transforms of the data. Then, $\tilde{d}_n^P(\boldsymbol{\omega}) = \tilde{C}_n^P(\boldsymbol{\omega}) + \iota\tilde{S}_n^P(\boldsymbol{\omega})$. The following result gives the asymptotic behavior of the DFTs in different cases treated in Theorems 3.1-3.3.

Theorem 3.4. *Suppose that* $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ *is a zero mean stationary random field satisfying Assumptions (A.1) and (A.2). Also, suppose that* $\{\boldsymbol{\omega}_{1n}\}, \ldots, \{\boldsymbol{\omega}_{rn}\}$ *are sequences satisfying (3.4).*

(a) If the limiting frequencies $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r$ *satisfy the conditions of Theorem 3.1, then*

$$[\tilde{C}_n^P(\boldsymbol{\omega}_{1n}), \tilde{S}_n^P(\boldsymbol{\omega}_{1n}), \ldots, \tilde{C}_n^P(\boldsymbol{\omega}_{rn}), \tilde{S}_n^P(\boldsymbol{\omega}_{rn})]$$

has the same limit distribution as that of the mean uncorrected version $[C_n^P(\boldsymbol{\omega}_{1n}), S_n^P(\boldsymbol{\omega}_{1n}), \ldots, C_n^P(\boldsymbol{\omega}_{rn}), S_n^P(\boldsymbol{\omega}_{rn})]$*, given by Theorem 3.1.*

(b) Suppose that $r = 2$ *and the sequences* $\{\boldsymbol{\omega}_{1n}\}, \{\boldsymbol{\omega}_{2n}\}$ *satisfy the conditions of one of the two parts of Theorem 3.2. Then,* $[\tilde{C}_n^P(\boldsymbol{\omega}_{1n}), \tilde{S}_n^P(\boldsymbol{\omega}_{1n}), \tilde{C}_n^P(\boldsymbol{\omega}_{2n}), \tilde{S}_n^P(\boldsymbol{\omega}_{2n})] \rightarrow^d N[\mathbf{0}, \Sigma]$*, with* Σ *as in the respective part of Theorem 3.2.*

(c) Suppose that $r = 2$ *and the sequences* $\{\boldsymbol{\omega}_{1n}\}, \{\boldsymbol{\omega}_{2n}\}$ *satisfy the conditions of any one of the three parts Theorem 3.3. Then,*

$$[\tilde{C}_n^P(\boldsymbol{\omega}_{1n}), \tilde{S}_n^P(\boldsymbol{\omega}_{1n}), \tilde{C}_n^P(\boldsymbol{\omega}_{2n}), \tilde{S}_n^P(\boldsymbol{\omega}_{2n})] \rightarrow^d N[\mathbf{0}, \tilde{\Sigma}],$$

where $\tilde{\Sigma} = \Sigma^i - \Sigma_0$ *for the* i*th part,* $i = (a), (b), (c)$ *and*

$$\Sigma_0 = \sigma_\infty^2 \begin{bmatrix} \tilde{\phi}_1^2(\mathbf{y}_1) & 2\tilde{\phi}_1(\mathbf{y}_1)\tilde{\phi}_2(\mathbf{y}_1) & 2\tilde{\phi}_1(\mathbf{y}_1)\tilde{\phi}_1(\mathbf{y}_2) & 2\tilde{\phi}_1(\mathbf{y}_1)\tilde{\phi}_2(\mathbf{y}_2) \\ & \tilde{\phi}_2^2(\mathbf{y}_1) & 2\tilde{\phi}_1(\mathbf{y}_2)\tilde{\phi}_2(\mathbf{y}_1) & 2\tilde{\phi}_2(\mathbf{y}_1)\tilde{\phi}_2(\mathbf{y}_2) \\ & & \tilde{\phi}_1^2(\mathbf{y}_2) & 2\tilde{\phi}_1(\mathbf{y}_2)\tilde{\phi}_2(\mathbf{y}_2) \\ & & & \tilde{\phi}_2^2(\mathbf{y}_2) \end{bmatrix}.$$

Thus, the asymptotic distributions of the sine and cosine transforms remain unchanged in all cases where the discrete Fourier frequencies converge to a nonzero limit. However, for frequency sequences converging to the zero frequency, the asymptotic covariance is different.

3.2 Results under the MID case

For the MID case, define the discrete Fourier transform (DFT) of $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_{N_n})\}$ by

$$d_n^M(\boldsymbol{\omega}) \quad = \quad N_n^{-1/2} \sum_{\mathbf{j}\in\mathbb{Z}^d:\Delta\mathbf{j}\eta_n\in\mathcal{D}_n} Z(\Delta\mathbf{j}\eta_n)\exp\left\{\iota\boldsymbol{\omega}'\Delta\mathbf{j}\eta_n\right\} \tag{3.14}$$

and the corresponding cosine and sine transforms $C_n^M(\boldsymbol{\omega})$ and $S_n^M(\boldsymbol{\omega})$ can be defined in a similar way. Although for each fixed n, the observations in the deterministic MID case lie on a grid, the asymptotic distribution of the DFT depends on the dependence structure of the random field $\{Z(\cdot)\}$ through the *full* spectral density function $\psi(\boldsymbol{\omega})$; knowledge of the folded spectral density ψ_Δ is no longer adequate as in the PID case. This is mainly due to the fact that the asymptotic variances of the relevant transforms in the MID case are given by certain integrals of the auto-covariance function over $\mathbb{R}^d$ as compared to infinite sums in the PID case. Further, the restriction on the limiting frequencies to lie in the set Π_Δ can be *dropped*, as it is now possible to infer about the full spectral density $\psi(\cdot)$ by considering the DFT at any given $\boldsymbol{\omega} \in \mathbb{R}^d$.

For the MID case, the following additional assumption will be used:

(A.3) $\lambda_n\eta_n \to \infty$.

The following result gives the asymptotic joint distribution of the sine and cosine transforms in the case of distinct *nonzero* limits.

Theorem 3.5. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1), (A.2) and (A.3). Also suppose that for $r \in \mathbb{N}$, $\{\boldsymbol{\omega}_{1n}\}, \ldots, \{\boldsymbol{\omega}_{rn}\}$ are frequency sequences of the form $\boldsymbol{\omega}_{jn} = 2\pi\mathbf{k}_{jn}/\lambda_n$ for $\mathbf{k}_{jn} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that $\mathbf{k}_{jn} \to \boldsymbol{\omega}_j \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and $\boldsymbol{\omega}_j \pm \boldsymbol{\omega}_k \neq \mathbf{0}$ for all $1 \leq j \neq k \leq r$. Then, $\eta_n^{d/2}[C_n^M(\boldsymbol{\omega}_{1n}), S_n^M(\boldsymbol{\omega}_{1n}), \cdots, C_n^M(\boldsymbol{\omega}_{rn}), S_n^M(\boldsymbol{\omega}_{rn})]' \xrightarrow{d} N[\mathbf{0}, \Sigma]$, where Σ is a block diagonal matrix with r blocks of the form $B_\ell I_2$ where, $B_\ell = \frac{1}{2}(\prod_{i=1}^n \delta_i)^{-1} (2\pi)^d\psi(\boldsymbol{\omega}_\ell)$ for $\ell = 1, \cdots, r$.*

Theorem 3.5 implies that for $\boldsymbol{\omega}_{1n}, \cdots, \boldsymbol{\omega}_{rn}$ converging to *different non-zero* limits, the corresponding DFTs are also asymptotically independent and the asymptotic variances depend on the spectral density function of the process at the limiting frequencies $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r$. Note that the conditions on the frequency sequences $\{\boldsymbol{\omega}_{jn}\}$ in Theorem 3.5 are weaker than those required in the PID case (cf. Theorem 3.1). Also note that the DFTs defined in (3.14) do not have a nondegenerate limit *unless* they are rescaled by the damping factor $\eta_n^{d/2}$. As the spacing of the grid goes to zero, observations at neighboring locations tend to have very strong correlations, and as a result, the variances of the sine and cosine transforms $S_n^M(\cdot)$ and $C_n^M(\cdot)$ grow at a rate faster than the sample size. As a result, the natural scaling by the inverse-square root of the sample size is not adequate under the

MID, and the additional multiplicative factor $\eta_n^{d/2}$ is needed to make the sine and cosine transforms converge to a nondegenerate normal limit.

Remark: Conclusions on the asymptotic independence of the DFTs do not change in the other scenarios covered by Theorems 3.2-3.4. Specifically, with the additional $\eta_n^{d/2}$ multiplicative factor, the sine- and the cosine- transforms in the MID case continue to have the same limits as their PID counterparts in the set ups of Theorems 3.2-3.4, where the folded spectral density ψ_Δ in the limit is replaced by ψ in every occurrence. The theorems for each of these cases are not restated to save space.

4 Results under the stochastic design

4.1 Definition of the DFT and some preliminaries

In the stochastic design case, the observations are given by $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n)\}$, under both the PID and the MID asymptotic structures. Thus, define the (scaled) DFT of the sample $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n)\}$ under the stochastic design as

$$\check{d}_n(\boldsymbol{\omega}) \quad = \quad \lambda_n^{d/2} n^{-1} \sum_{j=1}^{n} Z(\mathbf{s}_j) \exp\left(\iota \boldsymbol{\omega}' \mathbf{s}_j\right), \quad \boldsymbol{\omega} \in I\!R^d \tag{4.1}$$

for both PID and MID cases. Note that under both asymptotic structures, we use a common scaling $\lambda_n^{d/2}$, which is asymptotically equivalent to the square root of the sample size under the PID, but grows at a slower rate (than $n^{1/2}$) in the MID case. That this is the correct scaling sequence for a non-degenerate limit in both cases will be clear in the next section where the main results will be stated. In analogy to (4.1), also define the (scaled) cosine and the sine transforms of $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n)\}$ as

$$\begin{aligned} \check{C}_n(\boldsymbol{\omega}) \quad &= \quad \lambda_n^{d/2} n^{-1} \sum_{j=1}^{n} \cos(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \\ \check{S}_n(\boldsymbol{\omega}) \quad &= \quad \lambda_n^{d/2} n^{-1} \sum_{j=1}^{n} \sin(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \end{aligned} \tag{4.2}$$

$\boldsymbol{\omega} \in I\!R^d$. Then, $\check{d}_n(\boldsymbol{\omega}) = \check{C}_n(\boldsymbol{\omega}) + \iota \check{S}_n(\boldsymbol{\omega})$.

Note that under the stochastic design, the sampling sites are generated by a realization of the sequence $\{\mathbf{X}_n\}$. As a consequence, the distributions of the DFTs discussed in this section actually refer to their *conditional distribution* given $\{\mathbf{X}_n\}$ and the CLTs under the stochastic design assert *weak convergence of these conditional distributions to respective normal limits for almost all realizations of the sequence* $\{\mathbf{X}_n\}$ *under* $P_{\mathbf{X}}$, $P_{\mathbf{X}}$ denotes the joint distribution of the $\mathbf{X}_i$'s. Also, for brevity, *use the convention that* $(\infty)^{-1}a = 0$ *for all* $a \in I\!R$. Thus, in the statements of the theorems below, the condition $n/\lambda_n^d \to c_* \in (0, \infty]$, will cover both the cases, $c_* \in (0, \infty)$ for the

PID asymptotic structure and $c_* = \infty$ for the MID, in a unified way, and an expression of the form $c_*^{-1}a$ where $a \in I\!\!R$, will be interpreted as zero in the MID (i.e., $c_* = \infty$) case. Finally, set $K = \int f^2(\mathbf{x})d\mathbf{x}$ and $I_\psi = \int_{I\!\!R^d} \psi(\boldsymbol{\omega})d\boldsymbol{\omega}$.

4.2 Asymptotic distribution at nonzero frequencies

In this section, the asymptotic joint distribution of the sine and cosine transforms at a finite collection of frequencies $\boldsymbol{\omega}_{1n}, \cdots, \boldsymbol{\omega}_{rn}, 1 \le r < \infty$ is investigated, where

$$\boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j \in I\!\!R^d \text{ as } n \to \infty. \tag{4.3}$$

Since under the stochastic design the data-sites are randomly distributed, it is not required for the sequences $\{\boldsymbol{\omega}_{jn}\}$'s to satisfy (3.4). The first result concerns the asymptotic joint distribution at $\boldsymbol{\omega}_{1n}, \cdots, \boldsymbol{\omega}_{rn}$ in the case where $\pm\boldsymbol{\omega}_j$'s are distinct and nonzero.

Theorem 4.1. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in I\!\!R^d\}$ is a zero mean stationary random field satisfying (A.1), (A.2) and that it satisfies Assumption (A.4):*

(A.4) $\lambda_n \gg n^\epsilon$ for some $\epsilon > 0$ and $\lim_{n\to\infty} n/\lambda_n = c_* \in (0, \infty]$.

Also suppose that for $j = 1, \cdots, r$, $r \in I\!\!N$, $\{\boldsymbol{\omega}_{jn}\}$ are sequences satisfying $\boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j \in I\!\!R^d \setminus \{\mathbf{0}\}$ and $\boldsymbol{\omega}_j \pm \boldsymbol{\omega}_k \ne \mathbf{0}$ for all $1 \le j \ne k \le r$. Then, $[\check{C}_n(\boldsymbol{\omega}_{1n}), \check{S}_n(\boldsymbol{\omega}_{1n}), \cdots, \check{C}_n(\boldsymbol{\omega}_{rn}), \check{S}_n(\boldsymbol{\omega}_{rn})]' \xrightarrow{d} N[\mathbf{0}, \Sigma]$, a.s. $(P_{\mathbf{X}})$, where Σ is a block-diagonal matrix with r blocks of the form $\check{A}_j I_2$ with $2\check{A}_j = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\boldsymbol{\omega}_j)$ and where, $K = \int f^2(\mathbf{x})d\mathbf{x}$, $I_\psi = \int_{I\!\!R^d} \psi(\boldsymbol{\omega})d\boldsymbol{\omega}$.

As in the fixed design case, Theorem 4.1 implies that any collection of disjoint subsets of the $2r$ cosine and sine transforms are also asymptotically independent. However, the asymptotic distribution of the DFTs $(\check{d}_n(\boldsymbol{\omega}_{1n}), \cdots, \check{d}_n(\boldsymbol{\omega}_{rn}))$ under the stochastic design depends on three factors, namely, (i) on the dependence structure of the spatial process $\{Z(\cdot)\}$, through the spectral density $\psi(\cdot)$, (ii) on the design density $f(\cdot)$, through the constant K, and (iii) on the spatial asymptotic framework (PID *vs.* MID). Note that the asymptotic variance has a somewhat simpler form under the MID (i.e., $c_* = \infty$) case where the first term in $\check{A}_j$ drops out.

Next let us consider the case where the limit frequencies are not necessarily distinct. As before, to state the main results in a transparent manner, let us restrict our attention to the case $r = 2$; The conclusions can be generalized to the case $r > 2$ in an obvious manner. For $\mathbf{z} \in I\!\!R^d$, define

$$\begin{aligned} \int \cos(\mathbf{z}'\mathbf{x})f(\mathbf{x})d\mathbf{x} &= K_1(\mathbf{z}) \quad , \quad \int \sin(\mathbf{z}'\mathbf{x})f(\mathbf{x})d\mathbf{x} = K_2(\mathbf{z}), \\ \int \cos(\mathbf{z}'\mathbf{x})f^2(\mathbf{x})d\mathbf{x} &= K_3(\mathbf{z}) \quad , \quad \int \sin(\mathbf{z}'\mathbf{x})f^2(\mathbf{x})d\mathbf{x} = K_4(\mathbf{z}). \end{aligned}$$

Then the following results hold:

Theorem 4.2. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1), (A.2)and (A.4). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are two sequences satisfying (4.3) with $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega} \in I\!R^d \setminus \{\mathbf{0}\}$.*

(a) (Asymptotically distant frequencies): Suppose that $\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\| \to \infty$. Then,

$$[\check{C}_n(\boldsymbol{\omega}_{1n}), \check{S}_n(\boldsymbol{\omega}_{1n}), \check{C}_n(\boldsymbol{\omega}_{2n}), \check{S}_n(\boldsymbol{\omega}_{2n})]' \xrightarrow{d} N[\mathbf{0}, \Sigma] \quad a.s. \quad (P_{\mathbf{X}}). \tag{4.4}$$

where $\Sigma = \check{A}I_4$, with $2\check{A} = c_*^{-1}I_\psi + K \cdot (2\pi)^d \psi(\boldsymbol{\omega})$.

(b) (Asymptotically close frequencies): Suppose that

$$\left.\begin{array}{c} \lambda_n(\omega_{1n}^{(p)} - \omega_{2n}^{(p)}) \to z_p \quad as \quad n \to \infty \quad for\ all \ \ p = 1, \cdots, d \\ with \quad \mathbf{z} = (z_1, \ldots, z_d)' \in I\!R^d \setminus \{\mathbf{0}\}. \end{array}\right\} \tag{4.5}$$

Then, (4.4) holds with Σ has the form as in (3.8) with A_i's are replaced with $\check{A}_i$'s, for $i = 1, 2, 3$ where $2\check{A}_1 = c_*^{-1}I_\psi + K \cdot (2\pi)^d\psi(\boldsymbol{\omega})$, $2\check{A}_2 = c_*^{-1}I_\psi K_1(\mathbf{z}) + K_3(\mathbf{z})(2\pi)^d\psi(\boldsymbol{\omega})$, $2\check{A}_3 = c_*^{-1}I_\psi K_2(\mathbf{z}) + K_4(\mathbf{z})(2\pi)^d\psi(\boldsymbol{\omega})$.

Theorem 4.2 gives the asymptotic distribution of the DFTs when both frequency sequences converge to a common *non-zero* frequency. Note that the limiting covariance matrix for the asymptotically close frequencies depends on the spatial sampling density $f(\cdot)$ through all four functionals $K_1(\cdot)$-$K_4(\cdot)$, and the constant K. This typically makes the corresponding DFTs asymptotically dependent. In contrast, DFTs along *asymptotically distant* frequency sequences are asymptotically independent.

Next, let us consider the special case, where the sampling region $\mathcal{D}_0$ is of the form

$$\mathcal{D}_0 = (-a_1, b_1) \times \ldots \times (-a_d, b_d), \tag{4.6}$$

for some $0 < a_j, b_j \leq 1/2$, $j = 1, \ldots, d$ and the sampling density $f(\cdot)$ is *uniform* over $\mathcal{D}_0$. In this case, $K_i(\mathbf{z}) = 0$ *for all* $i = 1, \ldots, 4$, whenever $\mathbf{z}$ is of the form

$$\mathbf{z} = \left(\frac{2\pi\ell_1}{a_1 + b_1}, \ldots, \frac{2\pi\ell_d}{a_d + b_d}\right)', \tag{4.7}$$

for some $\ell_1, \ldots, \ell_d \in Z\!\!\!Z$ with $\sum_{j=1}^d |\ell_j| \neq 0$. As a result, asymptotic independence of the DFTs holds even for asymptotically close frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$, provided $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ satisfy (4.5) for some $\mathbf{z}$ of the form (4.7). The asymptotic independence property of the DFTs can be guaranteed *for all* distinct asymptotically close sequences if one restricts attention to DFTs based on frequency sequences of the form

$$\boldsymbol{\omega}_n = \left(\frac{2\pi k_{1n}}{a_1 + b_1}, \ldots, \frac{2\pi k_{dn}}{a_d + b_d}\right)', \tag{4.8}$$

with $k_{1n},\ldots,k_{dn}\in \mathbb{Z}$, $\sum_{j=1}^{d}|k_{jn}|\neq 0$.

Remark: It is worth noting that for a rectangular sampling region (with $\mathcal{D}_0$ as in (4.6)), a similar conclusion on asymptotic independence of the DFTs holds for a *more general* class of sampling densities that can be expressed as a convolution of a general probability distribution with a suitable uniform density. Specifically, the class of such sampling densities is given by $\mathcal{F}=\Big\{f: f$ has support $\mathcal{D}_0$, and $f(\cdot)=\int g_{\boldsymbol{\ell}}(\cdot-\mathbf{y})dG_{\boldsymbol{\ell}}(\mathbf{y})$ for some probability distribution $G_{\boldsymbol{\ell}}$ and for some $\boldsymbol{\ell}\in\mathcal{N}\Big\}$ where $g_{\boldsymbol{\ell}}$ is the density of the uniform distribution on $(-\frac{a_1}{\ell_1},\frac{b_1}{\ell_1})\times\ldots\times(-\frac{a_d}{\ell_d},\frac{b_d}{\ell_d})$ and $\mathcal{N}=\{(\ell_1,\ldots,\ell_d)'\in\mathbb{Z}^d:\ell_j\geq 2$ for all $j=1,\ldots,d\}$.

Remark: As in the deterministic case, there is a dual to Theorem 4.2, where $\boldsymbol{\omega}_1=-\boldsymbol{\omega}_2=\boldsymbol{\omega}\neq\mathbf{0}$. The formulation of the dual parallels that in the deterministic case with the A_j's in (3.9) replaced by $\check{A}_j$'s from Theorem 4.2.

4.3 Asymptotic distribution for the zero limiting frequency

Next consider the case where the fourier frequencies converge to the zero frequency. The asymptotic behavior of the cosine and sine transforms of spatial data under the PID and the MID asymptotic structures in the stochastic design case are given by the following theorem.

Theorem 4.3. *Suppose that $\{Z(\mathbf{s}):\mathbf{s}\in\mathbb{R}^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1), (A.2) and (A.4). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are two sequences in $\mathbb{R}^d\setminus\{\mathbf{0}\}$, converging to $\mathbf{0}$.*

(a) If $\|\lambda_n(\boldsymbol{\omega}_{1n}+\boldsymbol{\omega}_{2n})\|\to\infty$ and $\|\lambda_n(\boldsymbol{\omega}_{1n}-\boldsymbol{\omega}_{2n})\|\to\infty$, then

$$[\check{C}_n(\boldsymbol{\omega}_{1n}),\check{S}_n(\boldsymbol{\omega}_{1n}),\check{C}_n(\boldsymbol{\omega}_{2n}),\check{S}_n(\boldsymbol{\omega}_{2n})]'\xrightarrow{d}N\left[\mathbf{0},\Sigma^{(a)}\right],\quad a.s.\quad(P_{\mathbf{X}})\tag{4.9}$$

where $\Sigma^{(a)}=\check{A}^{[0]}I_4$, with $2\check{A}^{[0]}=c_*^{-1}I_\psi+K\cdot(2\pi)^d\psi(\mathbf{0})$.

(b) Suppose that $\|\lambda_n(\boldsymbol{\omega}_{1n}+\boldsymbol{\omega}_{2n})\|\to\infty$ but $\lambda_n(\boldsymbol{\omega}_{1n}-\boldsymbol{\omega}_{2n})\to\mathbf{z}_{12}\in\mathbb{R}^d$. Then, (4.9) holds with $\Sigma^{(a)}$ replaced by $\Sigma^{(b)}$, where $\Sigma^{(b)}$ has the form as in (3.8) with A_i's are replaced with $\check{A}_i^{[0]}$'s, for $i=1,2,3$ where $2\check{A}_1^{[0]}=c_*^{-1}I_\psi+K\cdot(2\pi)^d\psi(\mathbf{0})$, $2\check{A}_2^{[0]}=c_*^{-1}I_\psi K_1(\mathbf{z}_{12})+K_3(\mathbf{z}_{12})(2\pi)^d\psi(\mathbf{0})$, $2\check{A}_3^{[0]}=c_*^{-1}I_\psi K_2(\mathbf{z}_{12})+K_4(\mathbf{z}_{12})(2\pi)^d\psi(\mathbf{0})]$.

(c) Suppose that $\lambda_n(\boldsymbol{\omega}_{1n}+\boldsymbol{\omega}_{2n})\to\mathbf{y}_{12}\in\mathbb{R}^d$ and $\lambda_n(\boldsymbol{\omega}_{1n}-\boldsymbol{\omega}_{2n})\to\mathbf{z}_{12}\in\mathbb{R}^d$. Then, (4.9) holds with $\Sigma^{(a)}$ replaced by $\Sigma^{(c)}$, where the elements of $\Sigma^{(c)}$ are given by

$$\begin{aligned}
\Sigma_{11}^{(c)} &= (c_*^{-1}I_\psi/2)\{K_1(2\mathbf{y}_1)+1\}+(2\pi)^d(\psi(\mathbf{0})/2)\{K_3(2\mathbf{y}_1)+K\},\\
\Sigma_{12}^{(c)} &= c_*^{-1}I_\psi K_2(2\mathbf{y}_1)+(2\pi)^d\psi(\mathbf{0})K_4(2\mathbf{y}_1),\\
\Sigma_{13}^{(c)} &= c_*^{-1}I_\psi\{K_1(\mathbf{y}_{12})+K_1(\mathbf{z}_{12})\}+(2\pi)^d\psi(\mathbf{0})\{K_3(\mathbf{y}_{12})+K_3(\mathbf{z}_{12})\},\\
\Sigma_{14}^{(c)} &= c_*^{-1}I_\psi\{K_2(\mathbf{y}_{12})-K_2(\mathbf{z}_{12})\}+(2\pi)^d\psi(\mathbf{0})\{K_4(\mathbf{y}_{12})-K_4(\mathbf{z}_{12})\},
\end{aligned}$$

$$
\begin{aligned}
\Sigma_{22}^{(c)} &= (c_*^{-1} I_\psi/2)\{1 - K_1(2\mathbf{y}_1)\} + (2\pi)^d(\psi(\mathbf{0})/2)\{K - K_3(2\mathbf{y}_1)\}, \\
\Sigma_{23}^{(c)} &= c_*^{-1} I_\psi \{K_2(\mathbf{y}_{12}) + K_2(\mathbf{z}_{12})\} + (2\pi)^d \psi(\mathbf{0})\{K_4(\mathbf{y}_{12}) + K_4(\mathbf{z}_{12})\}, \\
\Sigma_{24}^{(c)} &= c_*^{-1} I_\psi \{K_1(\mathbf{z}_{12}) - K_1(\mathbf{y}_{12})\} + (2\pi)^d \psi(\mathbf{0})\{K_3(\mathbf{z}_{12}) - K_3(\mathbf{y}_{12})\}, \\
\Sigma_{33}^{(c)} &= (c_*^{-1} I_\psi/2)\{K_1(2\mathbf{y}_2) + 1\} + (2\pi)^d(\psi(\mathbf{0})/2)\{K_3(2\mathbf{y}_2) + K)\}, \\
\Sigma_{34}^{(c)} &= c_*^{-1} I_\psi K_2(2\mathbf{y}_2) + (2\pi)^d \psi(\mathbf{0}) K_4(2\mathbf{y}_2), \\
\Sigma_{44}^{(c)} &= (c_*^{-1} I_\psi/2)\{1 - K_1(2\mathbf{y}_2)\} + (2\pi)^d(\psi(\mathbf{0})/2)\{K - K_3(2\mathbf{y}_2)\},
\end{aligned}
$$

where $\mathbf{y}_1 = (\mathbf{y}_{12} + \mathbf{z}_{12})/2$ *and* $\mathbf{y}_2 = (\mathbf{y}_{12} - \mathbf{z}_{12})/2$.

As in Theorem 3.3, Theorem 4.3 shows that asymptotic independence of DFTs continues to hold for "asymptotically distant" Fourier frequency sequences converging to zero. For "asymptotically close" frequencies converging to zero, DFTs are typically asymptotically dependent. In the special case of the rectangular sampling region with a sampling design $f \in \mathcal{F}$, asymptotic independence of every pair of distinct sine and cosine transforms continues to hold, provided the frequency sequences are of the form (4.8).

Remark As in the last section, there is a dual to part (b) of Theorem 4.3, which is straightforward to formulate. Also, if $\boldsymbol{\omega}_n = \mathbf{0}$ for all $n \geq 1$, $\check{S}_n(\boldsymbol{\omega}_n) = 0$ for all $n \geq 1$, while $\check{C}_n(\boldsymbol{\omega}_n) = n^{-1/2}\sum_{i=1}^n Z(\mathbf{s}_i) \to^d N(0, c_*^{-1} I_\psi + K(2\pi)^d \psi(\mathbf{0})),\ \ a.s.\ (P_{\mathbf{X}})$. (cf. Lahiri (2003a)).

4.3.1 Results for mean-corrected DFTs

For stochastic design, the mean corrected DFT is defined as follows:

$$
\tilde{d}_n(\boldsymbol{\omega}) \quad \equiv \quad \lambda_n^{d/2} n^{-1} \sum_{j=1}^n [Z(\mathbf{s}_j) - \bar{Z}_n] \exp\left(\iota \boldsymbol{\omega}' \mathbf{s}_j\right), \quad \boldsymbol{\omega} \in I\!R^d \tag{4.10}
$$

where, $\bar{Z}_n = n^{-1}\sum_{j=1}^n Z(\mathbf{s}_j)$ is the sample mean. Similarly as before, define the mean corrected version of the cosine and the sine transforms of the data. The following result gives the asymptotic behavior of the DFTs in different cases treated in Theorems 4.1-4.3.

Theorem 4.4. *Suppose that* $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ *is a zero mean stationary random field satisfying (A.1), (A.2) and (A.4). Also, suppose that* $\{\boldsymbol{\omega}_{1n}\}, \ldots, \{\boldsymbol{\omega}_{rn}\}$ *are sequences satisfying (4.3).*

(a) If $\{\boldsymbol{\omega}_{1n}\}, \ldots, \{\boldsymbol{\omega}_{rn}\}$ *satisfy the conditions of Theorem 4.1, then* $[\tilde{C}_n(\boldsymbol{\omega}_{1n}), \tilde{S}_n(\boldsymbol{\omega}_{1n}), \ldots, \tilde{C}_n(\boldsymbol{\omega}_{rn}), \tilde{S}_n(\boldsymbol{\omega}_{rn})]$ *has the same limit distribution as that of the mean uncorrected version given by Theorem 4.1.*

(b) Suppose that the sequences $\{\boldsymbol{\omega}_{1n}\}$ *and* $\{\boldsymbol{\omega}_{2n}\}$ *satisfy the conditions of one of the two parts of Theorem 4.2. Then* $[\tilde{C}_n(\boldsymbol{\omega}_{1n}), \tilde{S}_n(\boldsymbol{\omega}_{1n}), \tilde{C}_n(\boldsymbol{\omega}_{2n}), \tilde{S}_n(\boldsymbol{\omega}_{2n})]$ *has the same limit distribution as its mean uncorrected version given by the respective part of Theorem 4.2.*

(c) Suppose that the sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ satisfy the conditions of any one of the three parts Theorem 4.3. Then,

$$[\tilde{C}_n(\boldsymbol{\omega}_{1n}), \tilde{S}_n(\boldsymbol{\omega}_{1n}), \tilde{C}_n(\boldsymbol{\omega}_{2n}), \tilde{S}_n(\boldsymbol{\omega}_{2n})] \to^d N[\mathbf{0}, \tilde{\Sigma}],\ a.s. (P_{\mathbf{X}})$$

where $\tilde{\Sigma} = \Sigma^i - \Sigma_0$ for the ith part, $i = (a), (b), (c)$ and

$$\Sigma_0 \quad = \quad \sigma_\infty^2 \begin{bmatrix} K_1^2(\mathbf{y}_1) & 2K_1(\mathbf{y}_1)K_2(\mathbf{y}_1) & 2K_1(\mathbf{y}_1)K_1(\mathbf{y}_2) & 2K_1(\mathbf{y}_1)K_2(\mathbf{y}_2) \\ & K_2^2(\mathbf{y}_1) & 2K_1(\mathbf{y}_2)K_2(\mathbf{y}_1) & 2K_2(\mathbf{y}_1)K_2(\mathbf{y}_2) \\ & & K_1^2(\mathbf{y}_2) & 2K_1(\mathbf{y}_2)K_2(\mathbf{y}_2) \\ & & & K_2^2(\mathbf{y}_2) \end{bmatrix},$$

with $2\sigma_\infty^2 = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\mathbf{0})$.

Thus, as in the fixed design case, the asymptotic distributions of the sine and cosine transforms with mean correction remain unchanged in all cases where the Fourier frequencies converge to a nonzero limit. However, for frequency sequences converging to the zero frequency, the asymptotic covariance can be different; the correction factor Σ_0 in the stochastic design case depends on the spatial sampling density $f(\cdot)$.

5 Some implications of the main results

The results on asymptotic joint distribution of the DFTs have some important implications for various frequency domain statistical inference methodologies. For example, the formulation of the frequency domain bootstrap (FDB) (cf. Franke and Härdle (1992)) critically depends on the asymptotic independence of the *full* set of DFTs. The results of Sections 3 and 4 show that for a sampling region of a general shape and/or for a general sampling density, the DFTs at asymptotically close ordinates are asymptotically correlated. As a result, formulation of the spatial version of the FDB must take into account the geometry of the sampling region under both the designs (i.e., fixed and stochastic) and, in addition, the non-uniformity of the sampling density in the stochastic design case.

As an example, consider the problem of non-parametric estimation of the spectral density and the auto-covariance function of the spatial process $\{Z(\cdot)\}$. In the regularly spaced data-sites case, the analogs of the standard time series formulas and bounds on the covariance between the DFTs (which is $O(n^{-1})$ where n is the sample size; See Priestley (1981)) no longer holds for a sampling region of a general shape, as the asymptotic correlation between neighboring DFTs do not vanish. As a result, consistency of the standard spectral density estimators based on kernel-smoothing of the sample periodograms need not hold. The situation gets worse in the case of irregularly spaced data-sites, as in this case, not only the geometry of the sampling region plays a crucial role, but the sampling density $f(\cdot)$ also has a nontrivial effect on the asymptotic distribution. For consistent estimation of the spectral density in the stochastic design case, one must also explicitly estimate various functionals of $f(\cdot)$ (e.g., the constant K, and the functions $K_1(\cdot), \ldots, K_4(\cdot)$) appearing in the asymptotic covariance matrix of the DFTs (cf. Theorems 4.2 and 4.3). Further, between the two asymptotic structures under the stochastic design case, the problem of estimating the spectral density and the auto-covariance function of the $Z(\cdot)$-process is *trickier* in the PID case. This is because under PID, $n/\lambda_n \to c_* \in (0, \infty)$ and the term $c_*^{-1} I_\psi$ in the asymptotic covariance matrices must be explicitly estimated. This observation has important implications for the popular nonparametric estimator of the auto-covariance function given by Hall and Patil (1994). Indeed, consistency of Hall and Patil (1994)'s estimator under the stochastic design is proved only under the MID asymptotic structure. Because of the presence of the extra term $c_*^{-1} I_\psi$, its consistency is no longer guaranteed under the PID case.

6 Proofs of the results from Section 3

6.1 Preliminaries

Let $\mathcal{U} = [0,1)^d$ denote the unit cube in $I\!R^d$. Let the autocovariance function of the stationary random field $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ be given by $\rho(\mathbf{s}) = cov(Z(\mathbf{s}), Z(\mathbf{0})), \mathbf{s} \in I\!R^d$. Then,

$$\begin{aligned} \rho(\mathbf{s}) &= \int_{I\!R^d} \exp(\iota \mathbf{s}' \boldsymbol{\omega}) \psi(\boldsymbol{\omega}) d\boldsymbol{\omega}, \ \mathbf{s} \in I\!R^d, \text{ and} \\ \rho(\Delta \mathbf{i}) &= \int_{\Pi_\Delta} \exp(\iota [\Delta \mathbf{i}]' \boldsymbol{\omega}) \psi_\Delta(\boldsymbol{\omega}) d\boldsymbol{\omega}, \ \mathbf{i} \in \mathbb{Z}^d. \end{aligned}$$

Let $\nu_\Delta(\cdot)$ denote the uniform distribution on $\Delta^{-1}\mathcal{D}_0$. Recall that the characteristic function of $\nu_\Delta(\cdot)$ is given by, $\int \exp(\iota \mathbf{t}' \mathbf{x}) d\nu_\Delta(d\mathbf{x}) = \phi_1(\mathbf{t}) + \iota \phi_2(\mathbf{t}), \ \mathbf{t} \in I\!R^d$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard basis of $I\!R^d$. Thus, $\mathbf{e}_1 = (1, 0, \ldots, 0)'$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)'$, etc. Let $C, C(\cdot)$ denote generic positive constants that depend on their arguments (if any), but not on n. Also, unless otherwise specified, limits in the order symbols are taken by letting $n \to \infty$.

The first lemma is a CLT for weighted sums of the form $\sum_{i=1}^n w_n(\mathbf{s}_i) Z(\mathbf{s}_i)$ with non-random weights under the deterministic spatial asymptotic framework as discussed earlier.

Lemma 6.1. *Let $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ be a zero mean stationary random field satisfying (A.1) and (A.2) and let $w_n(\cdot) : \mathcal{D}_n \to I\!R$ be a non-random bounded weight function. Suppose that there exists a function $Q : I\!R^d \to I\!R$ such that for any $\mathbf{h}_n \to \mathbf{h}$ in $I\!R^d$,*

$$\lim_{n \to \infty} c_n^{-2} \sum_{i: \mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n) = Q(\mathbf{h}). \tag{6.1}$$

where $c_n^2 = \sum_{i=1}^{N_n} w_n(\mathbf{s}_i)^2$. *Let* $\{\eta_n\}_{n\geq 1} \subset (0,\infty)$ *be a nonrandom sequence of real numbers such that* $\eta_n \equiv 1$ *for all* $n \geq 1$ *in the PID case and* $\eta_n \downarrow 0$ *as* $n \to \infty$ *in the MID case. Suppose that*

$$\max\{w_n^2(\mathbf{s}) : \mathbf{s} \in \mathcal{D}_n\}\lambda_n^d \eta_n^{-d} c_n^{-2} = O(1). \tag{6.2}$$

Then, with $N_n \equiv |\{\mathbf{j} : \mathbf{j} \in \mathbb{Z}^d, \eta_n \mathbf{j}/\lambda_n \in \Delta^{-1}\mathcal{D}_0\}|$,

$$\eta_n^{d/2} c_n^{-1} \sum_{i=1}^{N_n} w_n(\mathbf{s}_i) Z(\mathbf{s}_i) \ \xrightarrow{d} N(0, \sigma_\infty^2) \tag{6.3}$$

where $\sigma_\infty^2 = \sum_{\mathbf{i}\in\mathbb{Z}^d} \rho(\Delta\mathbf{i}) Q(\Delta\mathbf{i})$ *for the PID case and* $\sigma_\infty^2 = (\prod_{i=1}^d \delta_i)^{-1} \int_{I\!R^d} \rho(\mathbf{x}) Q(\mathbf{x}) d\mathbf{x}$ *for the MID case.*

Proof To prove the lemma, the conditions of Theorem 4.3 of Lahiri (2003a) for the PID case and Theorem 4.2 of Lahiri (2003a) for the MID case shall be verified. First note that under Assumption (A.1), since $\gamma_1(\cdot)$ is monotone (and left-continuous), there exists a $t_0 \in (0,\infty)$ such that

$$\gamma_1(t) \leq t^{-d(2+\tau)/\tau} \quad \text{for all} \quad t \geq t_0.$$

Write $\gamma_1^0(t) \equiv t^{-d(2+\tau)/\tau}$ for all $t > 0$. Since the weight function $w_n(\cdot)$ is bounded, by Remark 4.1 and Proposition 4.1 of Lahiri (2003a), it is enough to verify that

$$\gamma_2(t) = o\Big(\frac{[f_1^{-1}(t)]^d}{[t\gamma_1^0(t) f_1^{-1}(t)]}\Big) \quad \text{as} \quad t \to \infty, \tag{6.4}$$

where $f_1(t) = t^d \int_1^t y^{2d-1} \gamma_1^0(y) dy$, $t \in [1,\infty)$.

Write $\tau_0 = d(2+\tau)/\tau$. Then, $\tau_0 > 2d$ for $0 < \tau < 2$, $\tau_0 = 2d$ for $\tau = 2$, and $d < \tau_0 < 2d$ for $\tau \in (2,\infty)$. For $\tau \in (2,\infty)$, (6.4) follows from relation (4.4) of Lahiri (2003a) (with τ replaced by τ_0 and $a = 0$ therein). Next, check that ,

$$f_1(t) = Ct^d(1 + o(1)) \quad \text{as} \quad t \to \infty, \ \text{for } 0 < \tau < 2.$$

and,

$$f_1(t) = [t^d \log t](1 + o(1)) \quad \text{as} \quad t \to \infty, \ \text{for } \tau = 2.$$

Consider $\tau = 2$ first. In this case, $f_1^{-1}(t) = d^{1/d}[\frac{t}{\log t}]^{1/d}(1 + o(1))$ as $t \to \infty$ and hence,

$$[f_1^{-1}(t)]^d / [t\gamma_1^0(t) f_1^{-1}(t)] = t^2 [\log t]^{-3}(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

Hence, (6.4) holds. One can similarly establish (6.4) for $0 < \tau < 2$ using the growth rate of $f_1(\cdot)$ above, as in (4.5) of Lahiri (2003a). The lemma now follows from Theorems 4.2 and 4.3 of Lahiri (2003a).

Lemma 6.2. *Suppose that $\mathcal{D}_0$ be a Borel subset of $(-1/2, 1/2]^d$ such that the d-dimensional Lebesgue measure of its boundary $\partial\mathcal{D}_0$ is zero. Let $\{\eta_n\}_{n\geq 1}$ and N_n be as in Lemma 6.1. Also let $J_n = \{\mathbf{j} : \mathbf{j} \in \mathbb{Z}^d, \eta_n \mathbf{j}/\lambda_n \in \Delta^{-1}\mathcal{D}_0\}$ and $N_n = |J_n|$. Then*

(a) For any $K \in (0, \infty)$,

$$\sup_{\|\mathbf{z}\|\leq K} \left|\eta_n^d \lambda_n^{-d} \sum_{\mathbf{j}\in J_n} \exp(\iota 2\pi \mathbf{z}'\mathbf{j}\eta_n/\lambda_n) - \int_{\Delta^{-1}\mathcal{D}_0} \exp(\iota 2\pi \mathbf{z}'\mathbf{x}) d\mathbf{x}\right| \to 0 \quad as \quad n \to \infty.$$

*(b) (*A discrete version of the Reimann-Lebesgue lemma in d-dimension*). Let $\{\mathbf{z}_n\}_{n\geq 1} \subset \mathbb{R}^d$ be a sequence satisfying $\|\mathbf{z}_n\|^{-1} = o(1)$ and $\limsup_{n\to\infty} |\mathbf{e}_i'\mathbf{z}_n\eta_n|/\lambda_n < 1/2$ for all $i = 1, \ldots, d$. Then,*

$$\left|N_n^{-1} \sum_{\mathbf{j}\in J_n} \exp(\iota 2\pi \mathbf{z}_n'\mathbf{j}/\lambda_n)\right| \to 0 \quad as \quad n \to \infty.$$

Proof For simplicity, first consider the PID case. Then, $\eta_n \equiv 1$. Let $J_{1n} = \{\mathbf{j} : (\mathbf{j} + \mathcal{U})\lambda_n^{-1} \subset \Delta^{-1}\mathcal{D}_0\}$ and let $J_{2n} = J_n \setminus J_{1n}$. Also for $k = 1, 2$, write $\sum_k$ for summation over $\mathbf{j} \in J_{kn}$. Part (a) is a uniform version of the Riemann sum approximation to integrals and can be proved easily. Here an outline of the proof is given for completeness. For any $\mathbf{z} \in \mathbb{R}^d$ with $||\mathbf{z}|| \leq K$,

$$\begin{aligned}
&\left|\lambda_n^{-d} \sum_{\mathbf{j}\in J_n} \exp(\iota 2\pi \mathbf{z}'\mathbf{j}/\lambda_n) - \int_{\Delta^{-1}\mathcal{D}_0} \exp(\iota 2\pi \mathbf{z}'\mathbf{x}) d\mathbf{x}\right| \\
&\leq \sum\nolimits_1 \left|\int_{(\mathbf{j}+\mathcal{U})/\lambda_n} \left[\exp(\iota 2\pi \mathbf{z}'\mathbf{j}/\lambda_n) - \exp(\iota 2\pi \mathbf{z}'\mathbf{x})\right] d\mathbf{x}\right| \\
&\quad + \sum\nolimits_2 \left|\int_{[(\mathbf{j}+\mathcal{U})/\lambda_n]\cap\Delta^{-1}\mathcal{D}_0} \exp(\iota 2\pi \mathbf{z}'\mathbf{x}) d\mathbf{x}\right| + \lambda_n^{-d}|J_{2n}| \\
&\leq \sum\nolimits_1 \int_{(\mathbf{j}+\mathcal{U})/\lambda_n} (2\pi||\mathbf{z}||\sqrt{d}/\lambda_n) d\mathbf{x} + 2\lambda_n^{-d}|J_{2n}| \\
&\leq K(2\pi\sqrt{d})|J_{1n}|\lambda_n^{-d-1} + 2\lambda_n^{-d}|J_{2n}|.
\end{aligned}$$

Part (a) follows from this.

Nest consider part (b). Let $\phi(\cdot)$ denote the characteristic function of the uniform distribution on the unit cube $\mathcal{U}$. Then, it is easy to check that for any $\epsilon \in (0, 1/2)$,

$$\inf\{|\phi(\mathbf{t})| : \mathbf{t} \in [-\pi + \epsilon, \pi - \epsilon]^d\} \in (0, 1]. \tag{6.5}$$

Also,

$$\begin{aligned}
&\lambda_n^{-d} \sum_{\mathbf{j}\in J_{1n}} \exp(\iota 2\pi \mathbf{z}_n'\mathbf{j}/\lambda_n)\phi(2\pi\mathbf{z}_n/\lambda_n) \\
&= \lambda_n^{-d} \sum_{\mathbf{j}\in J_{1n}} \int_{\mathcal{U}} \exp(\iota 2\pi \mathbf{z}_n'[\mathbf{j} + \mathbf{x}]/\lambda_n) d\mathbf{x} \\
&= \sum_{\mathbf{j}\in J_{1n}} \int_{(\mathbf{j}+\mathcal{U})/\lambda_n} \exp(\iota 2\pi \mathbf{z}_n'\mathbf{x}) d\mathbf{x}.
\end{aligned} \tag{6.6}$$

Hence by (6.5) and (6.6) and the Riemann Lebesgue lemma, for any $\{\mathbf{z}_n\}_{n\geq 1}$ satisfying the conditions of part (b),

$$\begin{aligned}
&\left|\lambda_n^{-d}\sum_{\mathbf{j}\in J_n}\exp\left(\iota 2\pi\mathbf{z}_n^{'}\mathbf{j}/\lambda_n\right)\right| \\
&\leq \lambda_n^{-d}\left|\sum_{\mathbf{j}\in J_{1n}}\exp\left(\iota 2\pi\mathbf{z}_n^{'}\mathbf{j}/\lambda_n\right)\right| + |J_{2n}|\lambda_n^{-d} \\
&\leq \left|\phi(2\pi\mathbf{z}_n\lambda_n^{-1})\right|^{-1}\left\{\left|\int_{\Delta^{-1}\mathcal{D}_0}\exp\left(\iota 2\pi\mathbf{z}_n^{'}\mathbf{x}\right)d\mathbf{x}\right| + |J_{2n}|\lambda_n^{-d}\right\} + |J_{2n}|\lambda_n^{-d} \\
&= o(1).
\end{aligned}$$

In the MID case, both parts of Lemma 6.2 can be proved by retracing the above steps with λ_n replaced by $\eta_n^{-1}\lambda_n$ in every occurrence and hence the routine details have been omitted.

Remark Note that the conditions imposed on the boundary of $\mathcal{D}_0$ in Section 2.1 implies that the d-dimensional Lebesgue measure of its boundary $\partial\mathcal{D}_0$ is zero.

Remark For the MID case, analogs of parts (a) and (b) hold, provided in each appearance, λ_n is replaced by $\lambda_n\eta_n^{-1}$. Thus, J_n should be replaced by $J_n^{[1]} \equiv \{\mathbf{j}\in\mathbb{Z}^d : \eta\mathbf{j}/\lambda_n \in \Delta^{-1}\mathcal{D}_0\}$ and λ_n^{-d} in part (a) is replaced by $\lambda_n^{-d}\eta_n^d$. In addition, for part (b), N_n is now the sample size under MID.

6.2 Proof of Theorem 3.1

Fix $a_1,\cdots,a_r,b_1,\cdots,b_r \in \mathbb{R}$ with $\sum_{i=1}^r(a_i^2+b_i^2)\neq 0$. Then, we may write,

$$\begin{aligned}
&\sum_{p=1}^r[a_pC_n^P(\boldsymbol{\omega}_{pn}) + b_pS_n^P(\boldsymbol{\omega}_{pn})] \\
&= \sum_{\mathbf{j}\in J_n} Z(\Delta\mathbf{j})\left[N_n^{-1/2}\sum_{p=1}^r\{a_p\cos(\boldsymbol{\omega}_{pn}^{'}\Delta\mathbf{j}) + b_p\sin(\boldsymbol{\omega}_{pn}^{'}\Delta\mathbf{j})\}\right] \\
&= \sum_{\mathbf{j}\in J_n} Z(\Delta\mathbf{j})w_n(\Delta\mathbf{j}), \text{ (say).} \qquad (6.7)
\end{aligned}$$

Hence, to prove Theorem 3.1, it is enough to establish the asymptotic distribution of the weighted sum in (6.7). Note that, by (3.4), $\boldsymbol{\omega}_{jn}^{'}\Delta\mathbf{j} = 2\pi\lambda_n^{-1}\mathbf{k}_{jn}^{'}\mathbf{j}$ and hence,

$$\begin{aligned}
&\sum_{\mathbf{j}\in J_n} w_n^2(\Delta\mathbf{j}) \\
&= (2N_n)^{-1}\sum_{p=1}^r\sum_{q=1}^r\left[a_pa_q\sum_{\mathbf{j}\in J_n}\left\{\cos(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) + \cos(2\pi(\mathbf{k}_{pn}-\mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n)\right\}\right.
\end{aligned}$$

$$+a_p b_q \sum_{\mathbf{j}\in J_n} \Big\{\sin(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})'\mathbf{j}/\lambda_n) - \sin(2\pi(\mathbf{k}_{pn}-\mathbf{k}_{qn})'\mathbf{j}/\lambda_n)\Big\}$$
$$+a_q b_p \sum_{\mathbf{j}\in J_n} \Big\{\sin(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})'\mathbf{j}/\lambda_n) + \sin(2\pi(\mathbf{k}_{pn}-\mathbf{k}_{qn})'\mathbf{j}/\lambda_n)\Big\}$$
$$+b_p b_q \sum_{\mathbf{j}\in J_n} \Big\{\cos(2\pi(\mathbf{k}_{pn}-\mathbf{k}_{qn})'\mathbf{j}/\lambda_n) - \cos(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})'\mathbf{j}/\lambda_n)\Big\}\Bigg]. \tag{6.8}$$

Since under the conditions of Theorem 3.1, $2\pi(\mathbf{k}_{pn} \pm \mathbf{k}_{qn})/\lambda_n$ converges to a point in $\Pi^0_\Delta \setminus \{\mathbf{0}\}$ for all $1 \le p \ne q \le r$, by Lemma 6.2, part (b), it follows that

$$\sum_{\mathbf{j}\in J_n} w_n^2(\Delta\mathbf{j}) \to \frac{1}{2}\sum_{p=1}^{r}(a_p^2 + b_p^2).$$

Also, for any $\mathbf{h}_n \to \mathbf{h} \in I\!\!R^d$, by Lemma 6.2, part (a) and arguments similar to (6.8),

$$\lim_{n\to\infty} \sum_{i:\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}_n\in\mathcal{D}_n} w_n(\mathbf{s}_i)w_n(\mathbf{s}_i+\mathbf{h}_n) = \frac{1}{2}\sum_{p=1}^{r}(a_p^2+b_p^2)\cos(\boldsymbol{\omega}_p'\mathbf{h}). \tag{6.9}$$

Hence, by (6.8) and (6.9), condition (6.1) of Lemma 6.1 holds with

$$Q(\mathbf{h}) = [\sum_{p=1}^{r}(a_p^2+b_p^2)\cos(\boldsymbol{\omega}_p'\mathbf{h})]\Big/\sum_{p=1}^{r}(a_p^2+b_p^2).$$

Further, by (6.8) and the boundedness of $\cos(\cdot)$ and $\sin(\cdot)$, condition (6.2) of Lemma 6.1 holds. Next note that by the inversion formula,

$$\psi_\Delta(\boldsymbol{\omega}) = \frac{1}{vol(\Pi_\Delta)}\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho(\Delta\mathbf{j})\exp(-2\boldsymbol{\omega}'\Delta\mathbf{j}),\ \boldsymbol{\omega}\in\Pi_\Delta. \tag{6.10}$$

Hence, by Lemma 6.1, it follows that, $\sum_{p=1}^{r}[a_pC_n^P(\boldsymbol{\omega}_{pn}) + b_pS_n^P(\boldsymbol{\omega}_{pn})] \to^d N(0, \sum_{\mathbf{j}\in\mathbb{Z}^d}\rho(\Delta\mathbf{j})Q(\Delta\mathbf{j})$ $[(1/2)\sum_{p=1}^{r}(a_p^2+b_p^2)]) = N(0,(1/2)\sum_{p=1}^{r}(a_p^2+b_p^2)\psi_\Delta(\boldsymbol{\omega}_p)\,vol(\Pi_\Delta))$, completing the proof of Theorem 3.1.

6.3 Proof of Theorem 3.2

Fix $a_1, a_2, b_1, b_2 \in I\!\!R$ with $\sum_{i=1}^{2}(a_i^2+b_i^2) \ne 0$. Then, as in (6.7) we have,

$$\sum_{p=1}^{2}[a_pC_n^P(\boldsymbol{\omega}_{pn}) + b_pS_n^P(\boldsymbol{\omega}_{pn})] = \sum_{\mathbf{j}\in J_n} Z(\Delta\mathbf{j})w_n(\Delta\mathbf{j}), \tag{6.11}$$

where, $w_n(\mathbf{s}) = N_n^{-1/2}\sum_{p=1}^{2}\{a_p\cos(\boldsymbol{\omega}_{pn}'\mathbf{s}) + b_p\sin(\boldsymbol{\omega}_{pn}'\mathbf{s})\}$. Note that for any $\mathbf{h}_n \to \mathbf{h} \in I\!\!R^d$,

$$\sum_{i:\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}_n\in\mathcal{D}_n} w_n(\mathbf{s}_i)w_n(\mathbf{s}_i+\mathbf{h}_n)$$

$$= (2N_n)^{-1} \sum_{p=1}^{2} \sum_{q=1}^{2} \Bigg[a_p a_q \sum_{\mathbf{j} \in J_n} \Big\{ \cos(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})' \mathbf{j} \lambda_n^{-1} + \boldsymbol{\omega}'_{qn} \mathbf{h}_n) +$$

$$\cos(2\pi(\mathbf{k}_{pn} - \mathbf{k}_{qn})' \mathbf{j} \lambda_n^{-1} - \boldsymbol{\omega}'_{qn} \mathbf{h}_n) \Big\}$$

$$+ a_p b_q \sum_{\mathbf{j} \in J_n} \Big\{ \sin(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})' \mathbf{j} \lambda_n^{-1} + \boldsymbol{\omega}'_{qn} \mathbf{h}_n) + \sin(2\pi(\mathbf{k}_{qn} - \mathbf{k}_{pn})' \mathbf{j} \lambda_n^{-1} + \boldsymbol{\omega}'_{qn} \mathbf{h}_n) \Big\}$$

$$+ a_q b_p \sum_{\mathbf{j} \in J_n} \Big\{ \sin(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})' \mathbf{j} \lambda_n^{-1} + \boldsymbol{\omega}'_{qn} \mathbf{h}_n) - \sin(2\pi(\mathbf{k}_{qn} - \mathbf{k}_{pn})' \mathbf{j} \lambda_n^{-1} + \boldsymbol{\omega}'_{qn} \mathbf{h}_n) \Big\}$$

$$+ b_p b_q \sum_{\mathbf{j} \in J_n} \Big\{ \cos(2\pi(\mathbf{k}_{qn} - \mathbf{k}_{pn})' \mathbf{j} \lambda_n^{-1} + \boldsymbol{\omega}'_{qn} \mathbf{h}_n) - \cos(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})' \mathbf{j} \lambda_n^{-1} + \boldsymbol{\omega}'_{qn} \mathbf{h}_n) \Big\} \Bigg] \quad (6.12)$$

Note that, under the conditions of the theorem, for every pair $p, q \in \{1, 2\}$, $\{2\pi[\mathbf{k}_{pn} + \mathbf{k}_{qn}]\}$ converges to a non-zero element in $(-\pi/2, \pi/2)^d$. Thus, when the frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are asymptotically distant (cf.(3.5)), by Lemma 6.2(b),

$$\lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n) = \frac{1}{2} \sum_{p=1}^{2} (a_p^2 + b_p^2) \cos(\boldsymbol{\omega}' \mathbf{h}).$$

On the other hand, when $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ satisfy (3.6), by Lemma 6.2, (both parts (a) and (b)),

$$\lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n)$$
$$= \frac{1}{2} \sum_{p=1}^{2} \sum_{q=1}^{2} \Big[a_p a_q \int \cos(2\pi \boldsymbol{\ell}'_{pq} \mathbf{x} - \boldsymbol{\omega}' \mathbf{h}) d\nu_\Delta(\mathbf{x}) + a_p b_q \int \sin(2\pi \boldsymbol{\ell}'_{21} \mathbf{x} + \boldsymbol{\omega}' \mathbf{h}) d\nu_\Delta(\mathbf{x})$$
$$- a_q b_p \int \sin(2\pi \boldsymbol{\ell}'_{21} \mathbf{x} + \boldsymbol{\omega}' \mathbf{h}) d\nu_\Delta(\mathbf{x}) + b_p b_q \int \cos(2\pi \boldsymbol{\ell}'_{21} \mathbf{x} + \boldsymbol{\omega}' \mathbf{h}) d\nu_\Delta(\mathbf{x}) \Big],$$

where, $\boldsymbol{\ell}_{12} = \boldsymbol{\ell} = -\boldsymbol{\ell}_{21}$ and $\boldsymbol{\ell}_{11} = \boldsymbol{\ell}_{22} = 0$. The proof of Theorem 3.2 can now be completed using Lemma 6.1, the inversion formula (6.10) and the arguments in the proof of Theorem 3.1.

6.4 Proof of Theorem 3.3

Fix $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $\sum_{p=1}^{2} (a_p^2 + b_p^2) \neq 0$. Then, $\sum_{p=1}^{2} [a_p C_n^P(\boldsymbol{\omega}_{pn}) + b_p S_n^P(\boldsymbol{\omega}_{pn})]$ can be expressed in the from (6.11) and the corresponding weight function $w_n(\cdot)$ satisfies (6.12). For part (c), using Lemma 6.2 in (6.12), for any $\mathbf{h}_n \to \mathbf{h} \in \mathbb{R}^d$, we get

$$\lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n)$$
$$= \frac{1}{2} \sum_{p=1}^{2} \sum_{q=1}^{2} \Big[a_p a_q \Big\{ \int \cos(2\pi(\mathbf{k}_p + \mathbf{k}_q)' \mathbf{x}) d\nu_\Delta(\mathbf{x}) + \int \cos(2\pi(\mathbf{k}_p - \mathbf{k}_q)' \mathbf{x}) d\nu_\Delta(\mathbf{x}) \Big\}$$

$$+a_p b_q \left\{ \int \sin(2\pi(\mathbf{k}_p + \mathbf{k}_q)^{'}\mathbf{x}) d\nu_\Delta(\mathbf{x}) + \int \sin(2\pi(\mathbf{k}_q - \mathbf{k}_p)^{'}\mathbf{x}) d\nu_\Delta(\mathbf{x}) \right\}$$
$$+a_q b_p \left\{ \int \sin(2\pi(\mathbf{k}_p + \mathbf{k}_q)^{'}\mathbf{x}) d\nu_\Delta(\mathbf{x}) - \int \sin(2\pi(\mathbf{k}_q - \mathbf{k}_p)^{'}\mathbf{x}) d\nu_\Delta(\mathbf{x}) \right\}$$
$$+b_p b_q \left\{ \int \cos(2\pi(\mathbf{k}_q - \mathbf{k}_p)^{'}\mathbf{x}) d\nu_\Delta(\mathbf{x}) - \int \cos(2\pi(\mathbf{k}_p + \mathbf{k}_q)^{'}\mathbf{x}) d\nu_\Delta(\mathbf{x}) \right\} \Bigg],$$

where, $\mathbf{k}_i = \lim_{n\to\infty} [\lambda_n \Delta \boldsymbol{\omega}_{in}/2\pi]$, $i = 1, 2$ (which exist as $\lim_{n\to\infty} \lambda_n \boldsymbol{\omega}_{in}$ exist for both $i = 1, 2$ and are finite). The proof of part (c) now can be completed by applying the inversion formula (6.10) and the central limit theorem of Lemma 6.1. Parts (a) and (b) can be proved by repeating the steps in the proofs of parts (a) and (b) of Theorem 3.2 respectively, setting $\boldsymbol{\omega} = \mathbf{0}$. We omit the details to save space.

6.5 Proof of Theorem 3.4

Fix $a_1, \cdots, a_r, b_1, \cdots, b_r \in I\!R$ with $\sum_{i=1}^r (a_i^2 + b_i^2) \neq 0$. Note that,

$$\sum_{p=1}^r [a_p \tilde{C}_n^P(\boldsymbol{\omega}_{pn}) + b_p \tilde{S}_n^P(\boldsymbol{\omega}_{pn})] = \sum_{p=1}^r [a_p C_n^P(\boldsymbol{\omega}_{pn}) + b_p S_n^P(\boldsymbol{\omega}_{pn})] - N_n^{1/2} \bar{Z}_n \beta_n$$
$$= \sum_{i=1}^{N_n} Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - N_n^{-1/2}\beta_n] \tag{6.13}$$

where $w_n(\mathbf{s}) = N_n^{-1/2} \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}) + b_p \sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s})\}$ and $\beta_n = N_n^{-1} \sum_{i=1}^{N_n} \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}_i) + b_p \sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}_i)\}$. Now suppose, $\boldsymbol{\omega}_{pn} \to \boldsymbol{\omega} \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}$, for all $p = 1, 2, \cdots, r$. Then by Lemma 6.2(b), $\beta_n \to 0$. On the other hand, if $\lambda_n \boldsymbol{\omega}_{pn} \to \mathbf{y}_p \in I\!R^d$ for all $p = 1, 2, \cdots, r$, then by Lemma 6.2(b),

$$\beta_n \to \beta \equiv \sum_{p=1}^r \{a_p \int \cos(\mathbf{y}_p^{'}\Delta\mathbf{x}) d\nu_\Delta(\mathbf{x}) + b_p \int \sin(\mathbf{y}_p^{'}\Delta\mathbf{x}) d\nu_\Delta(\mathbf{x})\}. \tag{6.14}$$

For any $\mathbf{h}_n \to \mathbf{h} \in I\!R^d$, let $Q(\mathbf{h}) = \lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n)$. ¿From the proofs of theorems 3.1-3.3, it follows that under the hypothesis of each of the parts (a)-(c) of Theorem 3.4, $Q(\mathbf{h})$ exists. Hence, for any $\mathbf{h}_n \to \mathbf{h} \in I\!R^d$,

$$\lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} \left(w_n(\mathbf{s}_i) - N_n^{-1/2}\beta_n \right) \left(w_n(\mathbf{s}_i + \mathbf{h}_n) - N_n^{-1/2}\beta_n \right)$$
$$= \begin{cases} Q(\mathbf{h}) & : \ \boldsymbol{\omega}_{pn} \to \boldsymbol{\omega} \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}, \text{ for all } p = 1, \cdots, r, \\ Q(\mathbf{h}) - \beta^2 & : \ \boldsymbol{\omega}_{pn} \to \mathbf{0}, \text{ for all } p = 1, \cdots, r. \end{cases} \tag{6.15}$$

Note that,

$$\beta^2 = \sum_{p=1}^r \sum_{q=1}^r \Big[a_p a_q \tilde{\phi}_1(\mathbf{y}_p) \tilde{\phi}_1(\mathbf{y}_q) + a_p b_q \tilde{\phi}_1(\mathbf{y}_p) \tilde{\phi}_2(\mathbf{y}_q)$$
$$+ a_q b_p \tilde{\phi}_1(\mathbf{y}_q) \tilde{\phi}_2(\mathbf{y}_p) + b_p b_q \tilde{\phi}_2(\mathbf{y}_p) \tilde{\phi}_2(\mathbf{y}_q) \Big] \tag{6.16}$$

The result now follows from (6.13)-(6.16) and Lemma 6.1.

6.6 Proof of Theorem 3.5

The proof follows by retracing the above steps and employing Lemma 6.1 and 6.2 for the MID case along with the inversion formula,

$$\psi(\boldsymbol{\omega}) = (2\pi)^{-d} \int \rho(\mathbf{x}) \exp(-\iota\boldsymbol{\omega}'\mathbf{x})d\mathbf{x}, \;\; \boldsymbol{\omega} \in I\!R^d. \tag{6.17}$$

The routine details are omitted here.

7 Proofs of the results from Section 4

7.1 Preliminaries

Lemma 7.1. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ is a zero mean stationary random field satisfying (A.1), (A.2) and (A.4) and*

$$\max\{w_n^2(\mathbf{s}) : \mathbf{s} \in \mathcal{D}_n\} s_n^{-2} = O(1), \tag{7.1}$$

where $s_n^2 = Ew_n^2(\lambda_n \mathbf{X}_1)$. Also, suppose that there exists a function $Q_1(\cdot)$ such that for all $\mathbf{h} \in I\!R^d$,

$$\left[\int w^2(\lambda_n\mathbf{x})f(\mathbf{x})d\mathbf{x}\right]^{-1} \int w(\lambda_n\mathbf{x}+\mathbf{h})w(\lambda_n\mathbf{x})f^2(\mathbf{x})d\mathbf{x} \to Q_1(\mathbf{h}) \;\; as \; n \to \infty. \tag{7.2}$$

(i) If $n/\lambda_n^d \to c_* \in (0,\infty)$ as $n \to \infty$, then,

$$(ns_n^2)^{-1/2}\sum_{i=1}^n w_n(\mathbf{s}_i)Z(\mathbf{s}_i) \;\; \stackrel{d}{\longrightarrow} \;\; N\left(0, \rho(\mathbf{0}) + c_* \int_{I\!R^d} \rho(\mathbf{x})Q_1(\mathbf{x})d\mathbf{x}\right) \;\; a.s. \; (P_{\mathbf{X}}).$$

(ii) If $n/\lambda_n^d \to \infty$ as $n \to \infty$, then,

$$(n^2\lambda_n^{-d}s_n^2)^{-1/2}\sum_{i=1}^n w_n(\mathbf{s}_i)Z(\mathbf{s}_i) \;\; \stackrel{d}{\longrightarrow} \;\; N\left(0, \int_{I\!R^d} \rho(\mathbf{x})Q_1(\mathbf{x})d\mathbf{x}\right) \; a.s. \; (P_{\mathbf{X}}).$$

Proof : See Lahiri (2003a).

Lemma 7.2. *(*Multivariate Riemann-Lebesgue Lemma*): Let $f \in L^1(I\!R^d)$. Then,*

$$\lim_{\|\mathbf{t}\|\to\infty} \int_{I\!R^d} f(\mathbf{x})cos(\mathbf{t}'\mathbf{x})d\mathbf{x} \;\; = 0 = \;\; \lim_{\|\mathbf{t}\|\to\infty} \int_{I\!R^d} f(\mathbf{x})sin(\mathbf{t}'\mathbf{x})d\mathbf{x}.$$

Proof Follows by approximating f by a sequence of finite linear combinations of indicator functions of rectangular sets in $I\!R^d$ (which are dense in $L^1(I\!R^d)$) as in the one-dimensional case.

7.2 Proof of Theorem 4.1

Fix $a_1, \cdots, a_r, b_1, \cdots, b_r \in I\!R$ with $\sum_{p=1}^r (a_p^2 + b_p^2) \neq 0$. Let $\hat{C}_n(\boldsymbol{\omega}) = n^{-1/2} \sum_{j=1}^n \cos(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j)$ and $\hat{S}_n(\boldsymbol{\omega}) = n^{-1/2} \sum_{j=1}^n \sin(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j)$, $\boldsymbol{\omega} \in I\!R^d$. Note that

$$\sum_{p=1}^r [a_p \hat{C}_n(\boldsymbol{\omega}_{pn}) + b_p \hat{S}_n(\boldsymbol{\omega}_{pn})] = \sum_{j=1}^n Z(\mathbf{s}_j) w_n(\mathbf{s}_j), \tag{7.3}$$

where, $w_n(\mathbf{s}_j) = n^{-1/2} \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}' \mathbf{s}_j) + b_p \sin(\boldsymbol{\omega}_{pn}' \mathbf{s}_j)\}$. Proceeding similarly as (6.8), we may write

$$\begin{aligned}
&\int w_n^2(\lambda_n \mathbf{x}) f(\mathbf{x}) d\mathbf{x} \\
&= (2n)^{-1} \sum_{p=1}^r \sum_{q=1}^r \Big[a_p a_q \int \Big\{ \cos(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})' \mathbf{x}) + \cos(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})' \mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \\
&+ a_p b_q \int \Big\{ \sin(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})' \mathbf{x}) - \sin(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})' \mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \\
&+ a_q b_p \int \Big\{ \sin(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})' \mathbf{x}) + \sin(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})' \mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \\
&+ b_p b_q \int \Big\{ \cos(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})' \mathbf{x}) - \cos(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})' \mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \Big].
\end{aligned} \tag{7.4}$$

Hence under the conditions of Theorem 4.1 and Lemma 7.2, it follows that

$$n \int w_n^2(\lambda_n \mathbf{x}) f(\mathbf{x}) d\mathbf{x} \to \frac{1}{2} \sum_{p=1}^r (a_p^2 + b_p^2).$$

Also, for any $\mathbf{h} \in I\!R^d$, by Lemma 7.2 and arguments similar to (7.4),

$$\lim_{n \to \infty} n \int w_n(\lambda_n \mathbf{x} + \mathbf{h}) w_n(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} = \frac{K}{2} \sum_{p=1}^r (a_p^2 + b_p^2) \cos(\boldsymbol{\omega}_p' \mathbf{h}). \tag{7.5}$$

Hence, by (7.4) and (7.5), condition (7.2) holds with

$$Q_1(\mathbf{h}) = [K \sum_{p=1}^r (a_p^2 + b_p^2) \cos(\boldsymbol{\omega}_p' \mathbf{h})] \Big/ \sum_{p=1}^r (a_p^2 + b_p^2).$$

Further, by (7.4) and the boundedness of $\cos(\cdot)$ and $\sin(\cdot)$, condition (7.1) of Lemma 7.1 holds. Next by the inversion formula as in (6.17), it follows that

$$\begin{aligned}
&\sum_{p=1}^r [a_p \check{C}_n(\boldsymbol{\omega}_{pn}) + b_p \check{S}_n(\boldsymbol{\omega}_{pn})] \\
&\quad \to^d N \left(\mathbf{0}, (c_*^{-1} I_\psi / 2) \sum_{p=1}^r (a_p^2 + b_p^2) + (K/2)(2\pi)^d \sum_{p=1}^r \psi(\boldsymbol{\omega}_p)(a_p^2 + b_p^2) \right).
\end{aligned}$$

This completes the proof of Theorem 4.1.

7.3 Proof of Theorem 4.2

Fix $a_1, a_2, b_1, b_2 \in I\!R$ with $\sum_{i=1}^{2}(a_i^2 + b_i^2) \neq 0$. Then, as in (7.3) we have,

$$\sum_{p=1}^{2}[a_p\hat{C}_n(\boldsymbol{\omega}_{pn}) + b_p\hat{S}_n(\boldsymbol{\omega}_{pn})] = \sum_{j=1}^{n} Z(\mathbf{s}_j)w_n(\mathbf{s}_j), \tag{7.6}$$

where, $w_n(\mathbf{s}) = n^{-1/2}\sum_{p=1}^{2}\{a_p\cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}) + b_p\sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s})\}$. Then for any $\mathbf{h} \in I\!R^d$,

$$\begin{aligned}
&\int w_n(\lambda_n\mathbf{x} + \mathbf{h})w_n(\lambda_n\mathbf{x})f^2(\mathbf{x})d\mathbf{x} \\
&= (2n)^{-1}\sum_{p=1}^{2}\sum_{q=1}^{2}\Bigg[a_pa_q\int\Big\{\cos(\lambda_n(\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})^{'}\mathbf{x} + \boldsymbol{\omega}_{qn}^{'}\mathbf{h}) \\
&\quad +\cos(\lambda_n(\boldsymbol{\omega}_{qn} - \boldsymbol{\omega}_{pn})^{'}\mathbf{x} + \boldsymbol{\omega}_{qn}^{'}\mathbf{h})\Big\}f^2(\mathbf{x})d\mathbf{x} \\
&+a_pb_q\int\Big\{\sin(\lambda_n(\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})^{'}\mathbf{x} + \boldsymbol{\omega}_{qn}^{'}\mathbf{h}) + \sin(\lambda_n(\boldsymbol{\omega}_{qn} - \boldsymbol{\omega}_{pn})^{'}\mathbf{x} + \boldsymbol{\omega}_{qn}^{'}\mathbf{h})\Big\}f^2(\mathbf{x})d\mathbf{x} \\
&+a_qb_p\int\Big\{\sin(\lambda_n(\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})^{'}\mathbf{x} + \boldsymbol{\omega}_{qn}^{'}\mathbf{h}) - \sin(\lambda_n(\boldsymbol{\omega}_{qn} - \boldsymbol{\omega}_{pn})^{'}\mathbf{x} + \boldsymbol{\omega}_{qn}^{'}\mathbf{h})\Big\}f^2(\mathbf{x})d\mathbf{x} \\
&+b_pb_q\int\Big\{\cos(\lambda_n(\boldsymbol{\omega}_{qn} - \boldsymbol{\omega}_{pn})^{'}\mathbf{x} + \boldsymbol{\omega}_{pn}^{'}\mathbf{h}) - \cos(\lambda_n(\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})^{'}\mathbf{x} + \boldsymbol{\omega}_{qn}^{'}\mathbf{h})\Big\}f^2(\mathbf{x})d\mathbf{x}\Bigg].
\end{aligned} \tag{7.7}$$

When the frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are asymptotically distant, by Lemma 7.2,

$$\lim_{n\to\infty}\int w_n(\lambda_n\mathbf{x} + \mathbf{h})w_n(\lambda_n\mathbf{x})f^2(\mathbf{x})d\mathbf{x} = \frac{K}{2}\cos(\boldsymbol{\omega}^{'}\mathbf{h})\sum_{p=1}^{2}(a_p^2 + b_p^2).$$

On the other hand, when $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ satisfy (4.5), then by Lemma 7.2 and proceeding similarly as in Theorem 3.2, we get the required result.

7.4 Proof of Theorem 4.3

Fix $a_1, a_2, b_1, b_2 \in I\!R$ with $\sum_{p=1}^{2}(a_p^2+b_p^2) \neq 0$. Then, $\sum_{p=1}^{2}[a_p\hat{C}_n(\boldsymbol{\omega}_{pn})+b_p\hat{S}_n(\boldsymbol{\omega}_{pn})]$ can be expressed in the from (7.6) and the corresponding weight function $w_n(\cdot)$ satisfies (7.7). For part (c), for any $\mathbf{h} \in I\!R^d$, we get

$$\begin{aligned}
&\lim_{n\to\infty} n\int w_n(\lambda_n\mathbf{x} + \mathbf{h})w_n(\lambda_n\mathbf{x})f^2(\mathbf{x})d\mathbf{x} \\
&= \frac{1}{2}\sum_{p=1}^{2}\sum_{q=1}^{2}\Bigg[a_pa_q\left\{\int\cos((\mathbf{y}_p + \mathbf{y}_q)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) + \int\cos((\mathbf{y}_p - \mathbf{y}_q)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x})\right\} \\
&+a_pb_q\left\{\int\sin((\mathbf{y}_p + \mathbf{y}_q)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) + \int\sin((\mathbf{y}_q - \mathbf{y}_p)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x})\right\} \\
&+a_qb_p\left\{\int\sin((\mathbf{y}_p + \mathbf{y}_q)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) - \int\sin((\mathbf{y}_q - \mathbf{y}_p)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x})\right\} \\
&+b_pb_q\left\{\int\cos((\mathbf{y}_q - \mathbf{y}_p)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) - \int\cos(\mathbf{y}_p + \mathbf{y}_q)^{'}\mathbf{x})f^2(\mathbf{x})d(\mathbf{x})\right\}\Bigg],
\end{aligned}$$

where, $\mathbf{y}_i = \lim_{n\to\infty} \lambda_n \boldsymbol{\omega}_{in}$, $i = 1, 2$. The proof of part (c) now can be completed by applying the inversion formula (6.17) and the central limit theorem of Lemma 7.1. Parts (a) and (b) can be proved by repeating the steps in the proofs of parts (a) and (b) of Theorem 4.2, respectively, setting $\boldsymbol{\omega} = \mathbf{0}$. We omit the details to save space.

7.5 Proof of Theorem 4.4

Fix $a_1, \cdots, a_r, b_1, \cdots, b_r \in I\!R$ with $\sum_{i=1}^r (a_i^2 + b_i^2) \neq 0$. With an obvious definition of $\tilde{\hat{C}}_n(\boldsymbol{\omega})$ and $\tilde{\hat{S}}(\boldsymbol{\omega})$, we have

$$\sum_{p=1}^r [a_p \tilde{\hat{C}}_n(\boldsymbol{\omega}_{pn}) + b_p \tilde{\hat{S}}(\boldsymbol{\omega}_{pn})] = \sum_{i=1}^n Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - n^{-1/2}\beta_n] \tag{7.8}$$

where $w_n(\mathbf{s}) = n^{-1/2} \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}' \mathbf{s}) + b_p \sin(\boldsymbol{\omega}_{pn}' \mathbf{s})\}$ and $\beta_n = n^{-1} \sum_{i=1}^n \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}' \mathbf{s}_i) + b_p \sin(\boldsymbol{\omega}_{pn}' \mathbf{s}_i)\}$. Now suppose, $\boldsymbol{\omega}_{pn} \to \boldsymbol{\omega} \neq \mathbf{0}$, for all $p = 1, 2, \cdots, r$. Then by SLLN, $\beta_n \to^{a.s.} 0$. On the other hand, if $\lambda_n \boldsymbol{\omega}_{pn} \to \mathbf{y}_p \in I\!R^d$ for all $p = 1, 2, \cdots, r$, then,

$$\beta_n \to^{a.s.} \beta_0 \equiv \sum_{p=1}^r \{a_p \int \cos(\mathbf{y}_p' \mathbf{x}) f(\mathbf{x}) d(\mathbf{x}) + b_p \int \sin(\mathbf{y}_p' \mathbf{x}) f(\mathbf{x}) d(\mathbf{x})\}. \tag{7.9}$$

Now,

$$\sum_{p=1}^r [a_p \tilde{\hat{C}}_n(\boldsymbol{\omega}_{pn}) + b_p \tilde{\hat{S}}(\boldsymbol{\omega}_{pn})] = \sum_{i=1}^n Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - n^{-1/2}\beta_0] + R_n \tag{7.10}$$

where, $R_n = (\beta_n - \beta_0) n^{-1/2} \sum_{i=1}^n Z(\mathbf{s}_i) = o_p(1)$, $a.s.$ $(P_{\mathbf{X}})$. Therefore, it is enough to find the asymptotic distribution for $\sum_{i=1}^n Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - n^{-1/2}\beta_0]$. Let $w_n^\star(\mathbf{s}) = w_n(\mathbf{s}) - n^{-1/2}\beta_0$. Then,

$$n \int w_n^{\star 2}(\lambda_n \mathbf{x}) f(\mathbf{x}) d\mathbf{x} = n \int w_n^2(\lambda_n \mathbf{x}) f(\mathbf{x}) d\mathbf{x} - \beta_0^2 + o_p(1), \quad \text{a.s.} \quad (P_{\mathbf{X}}). \tag{7.11}$$

and,

$$\begin{aligned} & n \int w_n^\star(\lambda_n \mathbf{x} + \mathbf{h}) w_n^\star(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} \\ & = n \int w_n(\lambda_n \mathbf{x} + \mathbf{h}) w_n(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} - K\beta_0^2 + o_p(1), \quad \text{a.s.} \quad (P_{\mathbf{X}}). \end{aligned} \tag{7.12}$$

The results now follow from (7.8)-(7.12) and Lemma 7.1.
Acknowledgement The authors thank an anonymous referee for some constructive suggestions that improved the presentation of the paper.

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